

EQUICONVERGENCE OF SPECTRAL DECOMPOSITIONS OF HILL-SCHRÖDINGER OPERATORS

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ABSTRACT. We study in various functional spaces the equiconvergence of spectral decompositions of the Hill operator $L = -d^2/dx^2 + v(x)$, $x \in [0, \pi]$, with H_{per}^{-1} -potential and the free operator $L^0 = -d^2/dx^2$, subject to periodic, antiperiodic or Dirichlet boundary conditions.

In particular, we prove that

$$\|S_N - S_N^0 : L^a \rightarrow L^b\| \rightarrow 0 \quad \text{if } 1 < a \leq b < \infty, \quad 1/a - 1/b < 1/2,$$

where S_N and S_N^0 are the N -th partial sums of the spectral decompositions of L and L^0 . Moreover, if $v \in H^{-\alpha}$ with $1/2 < \alpha < 1$ and $\frac{1}{a} = \frac{3}{2} - \alpha$, then we obtain uniform equiconvergence: $\|S_N - S_N^0 : L^a \rightarrow L^\infty\| \rightarrow 0$ as $N \rightarrow \infty$.

Keywords: Hill-Schrödinger operators, singular potentials, spectral decompositions, equiconvergence.

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1. INTRODUCTION

1. Since the earlier days of the theory of eigenfunction expansions for ordinary differential operators (V. A. Steklov [48, 49], G. D. Birkhoff [2, 3], A. Haar [12]) one of a few central questions was the question about *equiconvergence* of eigenfunction expansions related to the same ordinary differential operator (o.d.o.) but subject to different Birkhoff-regular boundary conditions [30] or for o.d.o. with different coefficients but the same (or similar)

boundary conditions. To illustrate the problem let us remind two results of J. Tamarkin [52, 53, 54].

Let

$$(1.1) \quad l(y) = \frac{d^n y}{dx^n} + \sum_{k=0}^{n-2} p_k(x) y^k(x), \quad 0 \leq x \leq 1, \quad p_k \in L^1([0, 1]),$$

and n linearly independent *bc* (boundary conditions) which are *regular* (see [30]) define an operator L in $L^2([0, 1])$. Let $\Lambda = \{\lambda_j\}_1^\infty$ be the set of all eigenvalues of L , and $R(z) = (z - L)^{-1}$ be its resolvent. Define

$$(1.2) \quad S_r(f) = \frac{1}{2\pi i} \int_{C(r)} R(z) dz,$$

where

$$C(r) = \{z \in \mathbb{C} : |z| = r\}$$

with radii r chosen in such a way that

$$(1.3) \quad \text{dist}(C(r), \Lambda) \geq \varepsilon > 0,$$

and define the r -th "partial sum" of the trigonometric Fourier integral

$$(1.4) \quad \sigma_r(f) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\sin r(x - \xi)}{x - \xi} f(\xi) d\xi, \quad f \in L^1([0, 1]).$$

Claim 1 (J. Tamarkin [53, 54], M. Stone [50, 51]). *With notations (1.2)-(1.4) the following holds:*

$$(1.5) \quad \lim_{r \rightarrow \infty, r \in (1.3)} \|S_r(f) - \sigma_r(f)\|_{C(K)} = 0$$

for any compact K in $(0, 1)$.

Claim 2 (J. Tamarkin [53, 54]). *If L^0 is the free operator $\frac{d^n y}{dx^n}$, i.e., $p_k = 0$, then*

$$(1.6) \quad \lim_{r \rightarrow \infty, r \in (1.3)} \|S_r(f, L) - S_r(f, L^0)\|_{C[0, 1]} = 0 \quad \forall f \in L^1([0, 1]).$$

These two statements bring our attention to the distinction between equiconvergence *on compacts* inside of the open interval $(0, 1)$ (Claim 1) and *on the entire closed interval* $[0, 1]$ (Claim 2). Along the first line of research lately let us mention works of A. P. Khromov [20, 22, 23, 24], A. Minkin [27], V. S. Rykhlov [31, 32, 33, 34, 35, 36], A. M. Gomilko and G. V. Radzievskii [11], A. S. Lomov [25].

Quite exceptional is the paper [21] where a criterion of equiconvergence on the whole interval for two different Birkhoff-regular bvp and a given continuous function was found.

2. In the present paper we will focus on (*equi*)convergence on the whole interval in the case of o.d.o. of the second order, or Hill operators

$$(1.7) \quad L = -\frac{d^2 y}{dx^2} + v(x), \quad 0 \leq x \leq \pi,$$

with bc of three types:

- (a) *periodic* Per^+ : $y(0) = y(\pi), \quad y'(0) = y'(\pi);$
- (b) *anti-periodic* Per^- : $y(0) = -y(\pi), \quad y'(0) = -y'(\pi);$
- (c) *Dirichlet* Dir : $y(0) = 0, \quad y(\pi) = 0.$

By using the quasi-derivatives approach of A. Savchuk and A. Shkalikov [43, 45] (see also [44, 46, 47] and R. Hryniv and Ya. Mykytyuk [13]–[18]), we developed in [5, 6, 7] a Fourier method for studying the spectral properties of one dimensional Schrödinger operators with periodic complex-valued singular potentials of the form

$$(1.8) \quad v = Q', \quad Q \in L_{loc}^2(\mathbb{R}), \quad Q(x + \pi) = Q(x).$$

Following A. Savchuk and A. Shkalikov [43, 45], one may consider various boundary value problems on the interval $[0, \pi]$ in terms of quasi-derivative

$$y^{[1]} = y' - Qy.$$

In particular, the periodic and anti-periodic boundary conditions have the form

$$\begin{aligned} Per^+ : \quad & y(\pi) = y(0), \quad y^{[1]}(\pi) = y^{[1]}(0), \\ Per^- : \quad & y(\pi) = -y(0), \quad y^{[1]}(\pi) = -y^{[1]}(0). \end{aligned}$$

Of course, if Q is a continuous function, then Per^+ and Per^- coincide, respectively, with the classical periodic boundary condition (a) and (b). The Dirichlet boundary condition has the same form as in the classical case:

$$Dir : \quad y(\pi) = y(0) = 0.$$

For each of the boundary conditions $bc = Per^\pm, Dir$ the differential expression

$$\ell(y) = -(y^{[1]})' - Qy$$

gives a rise of a closed (self adjoint for real v) operator $L_{bc} = L_{bc}(v)$ in $H^0 = L^2([0, \pi])$, respectively, with a domain

$$(1.9) \quad D(L_{Per^\pm}) = \{y \in H^1 : y^{[1]} \in W_1^1([0, \pi]), \text{ } Per^\pm \text{ holds, } \ell(y) \in H^0\},$$

or

$$(1.10) \quad D(L_{Dir}) = \{y \in H^1 : y^{[1]} \in W_1^1([0, \pi]), \text{ } Dir \text{ holds, } \ell(y) \in H^0\}.$$

Let L_{bc}^0 denote the free operator $L^0 = -d^2/dx^2$ considered with boundary conditions bc . It is easy to describe the spectra and eigenfunctions of L_{bc}^0 for $bc = Per^\pm, Dir$:

(a) $Sp(L_{Per^+}^0) = \{n^2, n = 0, 2, 4, \dots\}$; its eigenspaces are $E_n^0 = Span\{e^{\pm inx}\}$ for $n > 0$ and $E_0^0 = \{const\}$, $\dim E_n^0 = 2$ for $n > 0$, and $\dim E_0^0 = 1$.

(b) $Sp(L_{Per^-}^0) = \{n^2, n = 1, 3, 5, \dots\}$; its eigenspaces are $E_n^0 = Span\{e^{\pm inx}\}$, and $\dim E_n^0 = 2$.

(c) $Sp(L_{Dir}^0) = \{n^2, n \in \mathbb{N}\}$; its eigenspaces are $E_n^0 = Span\{\sin nx\}$, and $\dim E_n^0 = 1$.

Depending on the boundary conditions, we consider as our canonical orthogonal normalized basis (o.n.b.) in $L^2([0, \pi])$ the system $u_k(x)$, $k \in \Gamma_{bc}$,

where

$$(1.11) \quad \text{if } bc = Per^+ \quad u_k = \exp(ikx), \quad k \in \Gamma_{Per^+} = 2\mathbb{Z};$$

$$(1.12) \quad \text{if } bc = Per^- \quad u_k = \exp(ikx), \quad k \in \Gamma_{Per^-} = 1 + 2\mathbb{Z};$$

$$(1.13) \quad \text{if } bc = Dir \quad u_k = \sqrt{2} \sin kx, \quad k \in \Gamma_{Dir} = \mathbb{N}.$$

Let us notice that $\{u_k(x), k \in \Gamma_{bc}\}$ is a complete system of unit eigenvectors of the operator L_{bc}^0 . They are uniformly bounded, namely

$$(1.14) \quad |u_k(x)| \leq \sqrt{2} \quad \forall k \in \Gamma_{bc}.$$

We set

$$(1.15) \quad H_{Per^\pm}^1 = \{f \in H^1 : f(\pi) = \pm f(0)\}, \quad H_{Dir}^1 = \{f \in H^1 : f(\pi) = f(0) = 0\}.$$

One can easily see that $\{e^{ikx}, k \in \Gamma_{Per^\pm}\}$ is an orthogonal basis in $H_{Per^\pm}^1$ and $\{\sqrt{2} \sin kx, k \in \mathbb{N}\}$ is an orthogonal basis in H_{Dir}^1 . From here it follows that

$$(1.16) \quad H_{bc}^1 = \left\{ f(x) = \sum_{k \in \Gamma_{bc}} f_k u_k(x) : \|f\|_{H^1}^2 = \sum_{k \in \Gamma_{bc}} (1 + k^2) |f_k|^2 < \infty \right\}.$$

The following proposition gives the Fourier representation of the operators L_{Per^\pm} and their domains (see [6, Prop.10]). Let v be a singular potential of the form (1.8) and let $Q(x) = \sum_{k \in 2\mathbb{Z}} q(k) e^{ikx}$ be the Fourier series of Q with respect to the orthonormal system $\{e^{ikx}, k \in 2\mathbb{Z}\}$. We set

$$(1.17) \quad V(k) = ik \cdot q(k), \quad k \in 2\mathbb{Z}.$$

Proposition 1. *In the above notations, if $y \in H_{Per^\pm}^1$, then we have $y = \sum_{\Gamma_{Per^\pm}} y_k e^{ikx} \in D(L_{Per^\pm})$ and $Ly = h = \sum_{\Gamma_{Per^\pm}} h_k e^{ikx} \in H^0$ if and only if*

$$(1.18) \quad h_k = h_k(y) := k^2 y_k + \sum_{m \in \Gamma_{Per^\pm}} V(k - m) y_m, \quad \sum |h_k|^2 < \infty,$$

i.e.,

$$(1.19) \quad D(L_{Per^\pm}) = \left\{ y \in H_{Per^\pm}^1 : (h_k(y))_{k \in \Gamma_{Per^\pm}} \in \ell^2(\Gamma_{Per^\pm}) \right\}$$

and

$$(1.20) \quad L_{Per^\pm}(y) = \sum_{k \in \Gamma_{Per^\pm}} h_k(y) e^{ikx}.$$

In the case of Dirichlet boundary conditions, we consider expansions about the o.n.b. $\{\sqrt{2} \sin kx, k \in \mathbb{N}\}$. Let

$$(1.21) \quad Q(x) = \sum_{k=1}^{\infty} \tilde{q}(k) \sqrt{2} \sin kx$$

be the sine Fourier expansion of Q . We set

$$(1.22) \quad \tilde{V}(0) = 0, \quad \tilde{V}(k) = k\tilde{q}(k) \quad \text{for } k \in \mathbb{N}.$$

Remark 2. *Since $v = Q'$, the function $Q(x)$ is defined up to a constant. The choice of this constant play no role in the case of periodic or antiperiodic boundary conditions – the coefficients $V(k)$ in (1.17) do not depend on such a choice. But in the case of Dirichlet boundary conditions the situation is different. Since*

$$\frac{1}{\pi} \int_0^\pi \sin mx \, dx = \begin{cases} 0 & \text{for even } m, \\ 2/m & \text{for odd } m, \end{cases}$$

if one add a constant C to $Q(x)$ then for odd m the coefficients $\tilde{V}(m)$ will change by $2C$.

If $v \in L^1([0, \pi])$ and $\int_0^\pi v(x)dx = 0$, then with $Q(x) = \int_0^x v(x)dx$ it follows that the numbers $\tilde{V}(k)$ are the Fourier coefficients of v about the o.n.b. $\{\sqrt{2}\cos kx, k \in \mathbb{Z}_+\}$. The following choice of constants guarantees that our formulas agree with the classical ones:

$$(1.23) \quad Q(0) = 0 \quad \text{if } Q \text{ is continuous at } 0.$$

Next we give the Fourier representation of the operators L_{Dir} and their domains (see [6, Prop.15]). Notice that the matrix of the operator L_{Dir} (see (1.24) below) does not depend on the choice of constants discussed in the above Remark.

Proposition 3. *In the above notations, if $y \in H_{Dir}^1$, then we have $y = \sum_{k=1}^\infty y_k \sqrt{2} \sin kx \in D(L_{Dir})$ and $Ly = h = \sum_{k=1}^\infty h_k(y) \sqrt{2} \sin kx \in H^0$ if and only if*

$$(1.24) \quad h_k(y) = k^2 y_k + \frac{1}{\sqrt{2}} \sum_{m=1}^\infty \left(\tilde{V}(|k-m|) - \tilde{V}(k+m) \right) y_m$$

and $\sum |h_k(y)|^2 < \infty$, i.e.,

$$(1.25) \quad D(L_{Dir}) = \left\{ y \in H_{Dir}^1 : (h_k(y))_1^\infty \in \ell^2(\mathbb{N}) \right\},$$

$$(1.26) \quad L_{Dir}(y) = \sum_{k=1}^\infty h_k(y) \sqrt{2} \sin kx.$$

3. We study the equiconvergence of spectral decompositions of the operators L_{Per^\pm}, L_{Dir} and, respectively, $L_{Per^\pm}^0, L_{Dir}^0$ by using their Fourier representations with respect to the corresponding o.n.b. (1.11)–(1.13). In view of Propositions 1 and 3, each of the operators $L = L_{Per^\pm}, L_{Dir}$ has the form

$$(1.27) \quad L = L^0 + V,$$

where the operators L^0 and V are defined, respectively, by their action on the sequence of Fourier coefficients of $y = \sum_{\Gamma_{Per^\pm}} y_k \exp ikx \in H_{Per^\pm}^1$ or $y = \sum_{\Gamma_{Dir}} y_k \sqrt{2} \sin kx \in H_{Dir}^1$ as follows:

$$(1.28) \quad L^0 : (y_k) \rightarrow (k^2 y_k), \quad k \in \Gamma_{Per^\pm}, \Gamma_{Dir},$$

$$(1.29) \quad V : (y_m) \rightarrow (z_k), \quad z_k = \sum_m V(k-m)y_m, \quad k, m \in \Gamma_{Per^\pm}$$

for $bc = Per^\pm$ with $V(k)$ given by (1.17), and

$$(1.30) \quad V : (y_m) \rightarrow (z_k), \quad z_k = \frac{1}{\sqrt{2}} \sum_{m=1}^{\infty} \left(\tilde{V}(|k-m|) - \tilde{V}(k+m) \right) y_m, \quad k, m \in \mathbb{N}$$

for $bc = Dir$ with $\tilde{V}(k)$ given by (1.22).

These matrix representations could be used (see [6, 7]) to justify the standard resolvent formula

$$(1.31) \quad R_\lambda = R_\lambda^0 + R_\lambda^0 V R_\lambda^0 + \sum_{m=2}^{\infty} R_\lambda^0 (V R_\lambda^0)^m, \quad \lambda \notin Sp(L_{bc}).$$

Let $\Pi_N = \Pi_N(\omega, h)$, $N \in \mathbb{N}$, $\omega, h > 0$ be the rectangle

$$(1.32) \quad \Pi_N = \{z = x + iy : -\omega \leq x \leq N^2 + N, \quad |y| \leq h\},$$

and let

$$(1.33) \quad S_N = \frac{1}{2\pi i} \int_{\partial \Pi_N} R_\lambda d\lambda, \quad S_N^0 = \frac{1}{2\pi i} \int_{\partial \Pi_N} R_\lambda^0 d\lambda.$$

The spectra of operators L_{Per^\pm} are discrete; there are numbers $\omega_0 = \omega_0(v)$, $h_0 = h_0(v)$ and $N_0 = N_0(v)$ such that for $\omega \geq \omega_0$, $h \geq h_0$ and $N \geq N_0$ the rectangle (1.32) contains all periodic, antiperiodic or Dirichlet eigenvalues which real part does not exceed $N^2 + N$ (see [6, 7]).

By (1.31) we have

$$(1.34) \quad S_N - S_N^0 = \frac{1}{2\pi i} \int_{\partial \Pi_N} (R_\lambda - R_\lambda^0) d\lambda = T_N + B_N,$$

where

$$(1.35) \quad T_N = \frac{1}{2\pi i} \int_{\partial \Pi_N} R_\lambda^0 V R_\lambda^0 d\lambda$$

$$(1.36) \quad B_N = \frac{1}{2\pi i} \int_{\partial \Pi_N} \sum_{m=2}^{\infty} R_\lambda^0 (V R_\lambda^0)^m d\lambda.$$

The representation (1.34) is crucial for our approach to equiconvergence in the case of singular potentials. The operator B_N gives the "easy" part $S_N - S_N^0$; we estimate from above its norm by integrating the norm of the integrand in (1.36). However, when estimating the norm of T_N it is essential

first to integrate over $\partial\Pi_N$ using residuum techniques – see Sections 3–5 for details.

4. How do the deviation-operators

$$(1.37) \quad S_N - S_N^0 : X \rightarrow Y, \quad N \rightarrow \infty,$$

behave for different pairs of functional Banach spaces?

How does this behavior depend on the potential v , or on parameters p or α if

$$(1.38) \quad v \in L^p, \quad 1 \leq p \leq 2, \quad \text{and} \quad v \in H^{-\alpha}, \quad 0 < \alpha \leq 1?$$

If $Y = \mathcal{C} = C([0, \pi])$ or $L^\infty([0, 1])$ in (1.37) we speak on *uniform equiconvergence*.

V. A. Marchenko [26] proved, in the case $bc = Dir$, that

$$(1.39) \quad \|(S_N - S_N^0)f\|_\infty \rightarrow 0$$

if $v \in L^1([0, \pi])$ and $f \in L^2([0, \pi])$. V. A. Vinokurov and V. A. Sadovnichii [55] showed (1.39) in the case when $bc = Dir$, v is real-valued such that

$$(1.40) \quad v = Q' \quad \text{with } Q \text{ being a periodic function of bounded variation, and } f \in L^1.$$

One of the main results of our paper is the following assertion (see Theorems 4 and 10)

Suppose v is complex-valued, and $bc = Dir, Per^+$ or Per^- . If $v \in L^1$ then

$$(1.41) \quad \|S_N - S_N^0 : L^1 \rightarrow \mathcal{C}\| \rightarrow 0.$$

If v satisfies (1.40), then

$$(1.42) \quad \|(S_N - S_N^0)f\|_\infty \rightarrow 0 \quad \forall f \in L^1([0, \pi]).$$

Notice that in this statement

- bc is not only Dir but Per^+ and Per^- as well;
- v is complex-valued;
- if $v \in L^1$ the claim (1.41) is made for norms of the deviation-operators.

(The latter means an improvement of Tamarkin's second theorem in the case of Hill operators as well.)

For the family of L^p -spaces we extend (1.39) to claim (see Theorem 5 and Corollary 6 in Sect. 2) the following:

Let $v \in L^p$, $1 \leq p \leq 2$, $1 \leq a \leq b \leq \infty$ and

$$(1.43) \quad 1/p + 1/a - 1/b < 2.$$

Then

$$(1.44) \quad \|S_N - S_N^0 : L^a \rightarrow L^b\| \leq C \|v\|_p N^{-\gamma},$$

where

$$(1.45) \quad \gamma = (1 - 1/p) + (1 - (1/a - 1/b)).$$

In Marchenko's case (1.39)

$$p = 1, a = 2, b = \infty \quad \text{so} \quad \gamma = 0 + (1 - 1/2) = 1/2;$$

therefore (1.44) and (1.45) imply that

$$(1.46) \quad \|S_N - S_N^0 : L^2 \rightarrow \mathcal{C}\| \leq C \|v\|_1 N^{-1/2}$$

For $bc = \text{Dir}$ I. V. Sadovnichaya [39, 40] considered the problem of *uniform equiconvergence* for Hill operators, respectively, with singular potentials $v \in H^{-\alpha}$, $1/2 < \alpha < 1$ and $v \in H^{-1}$; see related papers [37, 38, 41, 42] also.

We extend analysis to $bc = \text{Per}^+$ and Per^- and prove uniform equiconvergence as

$$(1.47) \quad \|S_N - S_N^0 : L^1 \rightarrow \mathcal{C}\| \rightarrow 0 \quad \text{if } v \in H^{-1/2},$$

and moreover, for $v \in H^{-\alpha}$, $\frac{1}{2} \leq \alpha < 1$ we have

$$(1.48) \quad \|S_N - S_N^0 : L^a \rightarrow \mathcal{C}\| \rightarrow 0, \quad 1/a = 3/2 - \alpha.$$

(See more precise and complete claims in Sections 3 and 4.) The cases $v \in H^{-1}$, $bc = \text{Per}^\pm$, remain unsolved although for $bc = \text{Dir}$ it has been successfully done by I. V. Sadovnichaya [40].

Remark. In our main statements on *uniform* equiconvergence (Theorems 4, 10, 14) the proofs give stronger claims on *absolute* convergence of Fourier coefficient sequences (f_k) , so the L^∞ -norms in the image-spaces could be changed to the Wiener norms $\|f\|_W = \sum |f_k|$. The inequality (1.14) guarantees that the Wiener norm is stronger than the L^∞ -norm.

5. Multidimensional analogs of the above questions are more complicated because the structure of the spectrum and eigenfunctions of the free operator, say, in the case of $L = -\Delta + v(x)$, $x \in G \subset \mathbb{R}^m$, G being a good bounded domain in \mathbb{R}^m , by itself is formidable problem – see for example [1]. It does not give any ready answers to be used in analysis of equiconvergence. Still let us mention [28] where one can find an example of multidimensional equiconvergence in the case of polyharmonic operators $(-\Delta)^a$ under strong assumptions on the dimension m and the order $2a$. Moreover, in the case of 1D Dirac operators L (see [4, 29]), when basic spectral properties of the free operator L^0 subject to periodic, antiperiodic or Dirichlet bc are well known, a series of statements on equiconvergence of spectral decompositions has been proven in [29]. In [8, 9], we considered the Dirac operator L subject to arbitrary regular bc . We constructed canonical Riesz bases of root functions of L^0 , used these bases to develop Fourier analysis of L , proved existence of Riesz type spectral decompositions of L and established for potentials $v \in H^\alpha$, $\alpha > 0$ uniform equiconvergence of the spectral decompositions of Dirac operators L and L^0 , subject to arbitrary regular bc .

The general approach and framework in this paper are similar to those in [29] (in the case $v \in L^p$, $1 < p \leq 2$) and [8, 9] (in the case $v \in H^{-1}$).

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2. THE CASE OF POTENTIALS $v \in L^p$, $p \in (1, 2]$.

1. First we consider the case of potentials $v \in L^p([0, \pi])$, $p \in (1, 2]$, which illustrates all crucial steps in our scheme but technically is more simple. We normalize the Lebesgue measure so that the interval $[0, \pi]$ has measure one, and set

$$\|f\|_p = \|f|L^p\| = \left(\frac{1}{\pi} \int_0^\pi |f(x)|^p \right)^{1/p}.$$

If F and G are two functions then we write $F \lesssim G$ for $x \in D$ (or simply $F \lesssim G$ when D is known by the context) if there is a constant $C > 0$ such that

$$F(x) \leq C \cdot G(x) \quad \forall x \in D.$$

We write $F \sim G$ if we have simultaneously $F \lesssim G$ and $G \lesssim F$.

Theorem 4. *Let a potential v be an L^p -function, $1 < p \leq 2$, $1/p + 1/q = 1$. Then, for $bc = \text{Per}^\pm$, Dir ,*

$$(2.1) \quad \|S_N - S_N^0 : L^1 \rightarrow \mathcal{C}\| \lesssim N^{-1/q}, \quad N \geq N_*(\|v\|_p).$$

Proof. By (1.32)–(1.36),

$$(2.2) \quad S_N - S_N^0 = \frac{1}{2\pi i} \int_{\partial \Pi_N} (R(z) - R^0(z)) dz,$$

where

$$(2.3) \quad R(z) - R^0(z) = \sum_{m=2}^{\infty} U(m),$$

and

$$(2.4) \quad U(1) = R^0, \quad U(k+1) = U(k)VR^0, \quad k \geq 1$$

with understanding that (2.2)–(2.4) hold if all the operators are well-defined and the series and integrals do converge. We justify the latter by using inequalities proven in Section 6, Appendix.

The following diagrams help:

$$(2.5) \quad \ell^r \xrightarrow{\mathcal{F}^{-1}} L^p \xrightarrow{V} L^r \xrightarrow{\mathcal{F}} \ell^p \xrightarrow{\tilde{R}^0} \ell^r, \quad D = \tilde{R}^0 \mathcal{F} V \mathcal{F}^{-1},$$

with r and ρ chosen so that

$$(2.6) \quad (a) \quad \frac{1}{r} + \frac{1}{\rho} = 1 \quad \text{and} \quad (b) \quad \frac{1}{\rho} + \frac{1}{p} = \frac{1}{r} \Rightarrow (c) \quad 1 > \frac{1}{r} = \frac{1}{2} \left(1 + \frac{1}{p} \right) > \frac{1}{2},$$

and, for $m \geq 2$,

$$(2.7) \quad L^1 \xrightarrow{\mathcal{F}} \ell^\infty \xrightarrow{\tilde{R}^0} \ell^r \xrightarrow{D^{m-2}} \ell^r \xrightarrow{\mathcal{F}^{-1}} L^\rho \xrightarrow{V} L^r \xrightarrow{\mathcal{F}} \ell^\rho \xrightarrow{\tilde{R}^0} \ell^1 \xrightarrow{\mathcal{F}^{-1}} \mathcal{C},$$

where

$$(2.8) \quad \mathcal{F} : f \rightarrow (\langle f, u_k \rangle)_{k \in \Gamma_{bc}}$$

puts in correspondence to f its sequence of Fourier coefficients with respect to the canonical o.n.b. (1.11)–(1.13),

$$\tilde{R}^0 : (t_k) \rightarrow \frac{t_k}{z - k^2} : k \in \Gamma_{bc}$$

is a multiplier-operator in sequence spaces, and

$$\mathcal{F}^{-1} : (t_k) \rightarrow \sum_{k \in \Gamma} t_k u_k(x)$$

is the restoration of a function from the sequence of its Fourier coefficients.

These are algebraic definitions but (2.6a) and (1.14) guarantee that

$$(2.9) \quad \|\mathcal{F} : L^r \rightarrow \ell^\rho\| \leq \sqrt{2}, \quad \|\mathcal{F}^{-1} : \ell^r \rightarrow L^\rho\| \leq \sqrt{2}$$

(see Hausdorff-Young Theorem, [56, Theorem XII.2.3]), and (2.6b) – with Hölder Inequality – shows that

$$(2.10) \quad \|V : L^\rho \rightarrow L^r\| = \|v\|_p.$$

Analogously, in the case of multiplier-operators $M : e_k \rightarrow m_k e_k$ in sequence spaces we have, for $1 \leq a < b \leq \infty$,

$$(2.11) \quad \|M : \ell^b \rightarrow \ell^a\| = \|(m_k)|\ell^c\| \quad \text{with} \quad 1/c + 1/b = 1/a.$$

Now Diagram (2.5) shows that

$$(2.12) \quad \|D : \ell^r \rightarrow \ell^r\| = \|\tilde{R}^0 \mathcal{F} V \mathcal{F}^{-1} : \ell^r \rightarrow \ell^r\| \leq 2 \|\tilde{R}^0| \ell^p\| \cdot \|v\|_p.$$

Diagram (2.7) gives a factorization of the operator $U(m)$, $m \geq 2$, so we obtain

$$(2.13) \quad \|U(m) : L^1 \rightarrow \mathcal{C}\| \leq 4 \|\tilde{R}^0| \ell^r\| \cdot \|D\|^{m-2} \cdot \|\tilde{R}^0| \ell^r\| \cdot \|v\|_p.$$

Next we will use (2.12), (2.13) and inequalities from Appendix to get estimates for the norms of operators (2.2)–(2.3). The horizontal sides and left vertical side of $\partial \Pi_N$ could be sent to infinity (see Appendix, Lemmas 26 and 27), so

$$(2.14) \quad S_N - S_N^0 = \frac{1}{2\pi i} \int_{\Lambda_N} \sum_{m=2}^{\infty} U(m) dy$$

with

$$(2.15) \quad \Lambda_N = \{z = N^2 + N + iy, y \in \mathbb{R}\}$$

if we succeed to get good norm estimates on the line Λ_N .

Notice that in (2.12), for $z \in \Lambda_N$,

$$(2.16) \quad \|\tilde{R}^0|_{\ell^p}\| = A(z; p) = \left(\sum_{k \in \Gamma_{bc}} \frac{1}{|z - k^2|^p} \right)^{1/p}, \quad z = N^2 + N + iy,$$

so we obtain, by Inequality (6.26),

$$(2.17) \quad \|\tilde{R}^0|_{\ell^p}\| \leq C(p)N^{-1}.$$

Now (2.12) and (2.17) imply that there is $N_* = N_*(\|v\|_p)$ such that for $N \geq N_*$ we have

$$(2.18) \quad \|D\| \leq 2C(p)N^{-1}\|v\|_p < 1/4, \quad z = N^2 + N + iy.$$

Thus, for $z = N^2 + N + iy$ with $N \geq N_*$, it follows from (2.13) that

$$\|U(m) : L^1 \rightarrow \mathcal{C}\| \leq 4^{3-m} \|\tilde{R}^0|_{\ell^r}\|^2 \|v\|_p = 4^{3-m} A^2(z; r) \|v\|_p,$$

so by (2.14)

$$(2.19) \quad \|S_N - S_N^0 : L^1 \rightarrow \mathcal{C}\| \leq 8\|v\|_p \int_{\Lambda_N} A^2(z; r) dy.$$

Corollary 30, the third line in (6.31), asserts that

$$(2.20) \quad \int_{\Lambda} A^2(z; r) dy \leq C(r)N^{-2(1-1/r)}.$$

By (2.6c), we have

$$(2.21) \quad 2(1 - 1/r) = 2 - (1 + 1/p) = 1/q.$$

Therefore, (2.19), (2.20) and (2.21) imply (2.1), which completes the proof. \square

2. Next we estimate the norms $\|S_N - S_N^0 : L^a \rightarrow L^b\|$, where L^a and L^b are intermediate spaces such that

$$(2.22) \quad L^1 \supset L^a \supset L^b \supset L^\infty, \quad 1 \leq a \leq 2 \leq b \leq \infty \text{ with } 1/a - 1/b < 1.$$

Now we consider the following factoring of $U(m)$:

$$(2.23) \quad L^a \xrightarrow{\mathcal{F}} \ell^\alpha \xrightarrow{\tilde{R}^0} \ell^\tau \xrightarrow{D^{m-2}} \ell^\tau \xrightarrow{\mathcal{F}^{-1}} L^t \xrightarrow{V} L^s \xrightarrow{\mathcal{F}} \ell^\sigma \xrightarrow{\tilde{R}^0} \ell^\beta \xrightarrow{\mathcal{F}^{-1}} L^\beta,$$

where the operator D is defined from the diagram

$$(2.24) \quad \ell^\tau \xrightarrow{\mathcal{F}^{-1}} L^t \xrightarrow{V} L^s \xrightarrow{\mathcal{F}} \ell^\sigma \xrightarrow{\tilde{R}^0} \ell^\tau, \quad D = \tilde{R}^0 \mathcal{F} V \mathcal{F}^{-1}.$$

The arrow-operators in the above diagrams act as bounded operators between the corresponding Banach spaces (of functions or sequences) if the following seven conditions hold:

$$(C1) \quad 1/a + 1/\alpha = 1, \quad a \leq 2 \text{ (Hausdorff-Young);}$$

(C2) $1/\alpha + 1/r = 1/\tau$, (Hölder inequality) with r chosen to measure the norm of the multiplier-operator \tilde{R}^0 ;

$$(C3) \quad 1/\tau + 1/t = 1, \quad \tau \leq 2 \text{ (Hausdorff-Young);}$$

$$(C4) \quad 1/t + 1/p = 1/s, \text{ (Hölder inequality) with } v \in L^p;$$

$$(C5) \quad 1/s + 1/\sigma = 1, \quad s \leq 2 \text{ (Hausdorff-Young);}$$

$$(C6) \quad 1/\sigma + 1/r = 1/\beta, \text{ (Hölder inequality);}$$

$$(C7) \quad 1/\beta + 1/b = 1, \quad \beta \leq 2 \text{ (Hausdorff-Young).}$$

One can easily see that (C1)–(C7) imply (together with (2.22))

$$(2.25) \quad 1/r = \frac{1}{2}(1/a - 1/b + 1/p) < 1, \quad p \in [1, 2].$$

Moreover, if r is given by (2.25), then the parameters τ, t, s, σ are uniquely determined by (C2)–(C5), and we have $\tau \leq 2, s \leq 2$.

As in the proof of Theorem 4 we want to use (2.2)–(2.3) and prove that the series on the right side below converges and

$$(2.26) \quad S_N - S_N^0 = \frac{1}{2\pi} \int_{\Lambda_N} \sum_{m=2}^{\infty} U(m) dy.$$

Since $\|\tilde{R}^0 : \ell^\sigma \rightarrow \ell^\tau\| = \|\tilde{R}^0| \ell^p\|$, from (2.24) and (2.16) it follows that

$$(2.27) \quad \|D : \ell^\tau \rightarrow \ell^\tau\| \leq \|\tilde{R}^0| \ell^p\| \cdot \|v\|_p = A(z; p) \cdot \|v\|_p, \quad z \in \Lambda_N.$$

Lemma 29, (6.25) and (6.26) in Appendix show that even in the worst case, for $p \geq 1$,

$$(2.28) \quad \|\tilde{R}^0| \ell^p\| = A(z; p) \lesssim \frac{\log N}{N}, \quad z \in \Lambda_N.$$

Therefore, in view of (2.27) and (2.28), there is N_* such that

$$(2.29) \quad \|D : \ell^\tau \rightarrow \ell^\tau\| < 1/2, \quad N \geq N_*, \quad z \in \Lambda_N.$$

We have chosen r so that the norms in (C2) and (C6) are equal:

$$(2.30) \quad \|\tilde{R}^0 : \ell^\alpha \rightarrow \ell^\tau\| = \|\tilde{R}^0 : \ell^\sigma \rightarrow \ell^\beta\| = \|\tilde{R}^0| \ell^r\|.$$

Now (2.23) together with (2.25)–(2.30) show that

$$\|U(m) : L^a \rightarrow L^b\| \leq (1/2)^{m-2} \|v\|_p \|\tilde{R}^0| \ell^r\|^2 \quad \text{for } N \geq N_*, \quad z \in \Lambda_N.$$

Therefore, by (2.16) and (2.26),

$$(2.31) \quad \|S_N - S_N^0 : L^a \rightarrow L^b\| \leq \int_{\Lambda_N} \sum_{m=2}^{\infty} \|U(m)\| dy \leq 2\|v\|_p \int_{\Lambda_N} A^2(z; r) dy.$$

Corollary 30, see Appendix, gives estimates for $\int_{\Lambda} A^2(z; r) dy$. In view of (2.25), it leads us to the following.

Theorem 5. *Suppose $v \in L^p$, $1 \leq p \leq 2$ and $bc = Per^{\pm}$ or $bc = Dir$. If (2.22) holds, then $1/r = \frac{1}{2}(1/p + 1/a - 1/b) < 1$ and for $N \geq N_*(\|v\|_p)$*

$$(2.32) \quad \|S_N - S_N^0 : L^a \rightarrow L^b\| \lesssim \begin{cases} \frac{1}{N} & \text{if } r > 2; \\ \frac{\log N}{N} & \text{if } r = 2; \\ N^{-\gamma} & \text{if } r < 2, \end{cases}$$

with

$$(2.33) \quad \gamma = (1 - 1/p) + (1 - (1/a - 1/b)).$$

Corollary 6. *If $1 < a \leq 2$, $b = \infty$, and $1 \leq p < 2$ then $1/r > 1/2$ and by (2.32) and (2.33)*

$$(2.34) \quad \|S_N - S_N^0 : L^a \rightarrow \mathcal{C}\| \lesssim N^{-\delta}, \quad \delta = (1 - 1/p) + (1 - 1/a).$$

3. L^1 -POTENTIALS AND WEAKLY SINGULAR POTENTIALS

1. Now we consider the uniform equiconvergence for functions in L^1 in the case of L^1 -potentials and potentials v which are derivatives of functions of bounded variation.

J. Tamarkin [53, 54] proved – even in the more general case of higher order ordinary differential operators subject to Birkhoff-regular boundary conditions – that

$$(3.1) \quad \text{if } v, f \in L^1 \text{ then } \|(S_N - S_N^0)f\|_{C[0, \pi]} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

We will show, for $bc = Per^{\pm}$ and $bc = Dir$, that not only strong convergence holds but the norm convergence to zero of the deviation operators $S_N - S_N^0$ takes place as well.

2. Let $v \in L^1$, $bc = Per^{\pm}$ or Dir . We set

$$(3.2) \quad v^*(n) = \begin{cases} \sup\{V(k) : |k| \geq n\} & \text{if } bc = Per^{\pm}, \\ \sup\{\tilde{V}(k) : k \geq n\} & \text{if } bc = Dir, \end{cases}$$

where $V(k)$, $k \in 2\mathbb{Z}$ and $\tilde{V}(k)$, $k \in \mathbb{N}$ are, respectively, the Fourier coefficients of $v(x)$ about the systems $\{e^{ikx}, k \in 2\mathbb{Z}\}$ and $\{\sqrt{2} \cos kx, k \in \mathbb{N}\}$.

Theorem 7. *If $v \in L^1$ then, for $0 < H \leq N$,*

$$(3.3) \quad \|S_N - S_N^0 : L^1 \rightarrow \mathcal{C}\| \lesssim v^*(H) + \frac{H}{N}.$$

In particular, if $H = N^{\gamma}$, $0 < \gamma < 1$, then we have

$$(3.4) \quad \|S_N - S_N^0 : L^1 \rightarrow \mathcal{C}\| \lesssim v^*(N^{\gamma}) + N^{\gamma-1}.$$

Proof. By (1.34), $S_N - S_N^0 = T_N + B_N$, where T_N and B_N are given, respectively, by (1.35) and (1.36).

In Formula (1.36), the horizontal sides and left vertical side of $\partial\Pi_N$ could be sent to infinity (see Appendix, Lemmas 26 and 27), so it follows that

$$(3.5) \quad B_N = \frac{1}{2\pi i} \int_{\Lambda_N} \sum_{m=3}^{\infty} U(m) dy.$$

As in Section 2, we analyze the series under the integral in (3.5) by using diagrams. We factor the operator $U(m)$, $m \geq 1$, by the diagram

$$(3.6) \quad L^1 \xrightarrow{\mathcal{F}} \ell^\infty \xrightarrow{\tilde{R}^0} \ell^1 \xrightarrow{D^{m-1}} \ell^1 \xrightarrow{\mathcal{F}^{-1}} \mathcal{C}, \quad U(m) = \mathcal{F}^{-1} D^{m-1} \tilde{R}^0 \mathcal{F},$$

where the operator $D : \ell^1 \rightarrow \ell^1$ is defined by

$$(3.7) \quad D = \tilde{R}^0 \mathcal{F} V \mathcal{F}^{-1}, \quad \ell^1 \xrightarrow{\mathcal{F}^{-1}} L^\infty \xrightarrow{V} L^1 \xrightarrow{\mathcal{F}} \ell^\infty \xrightarrow{\tilde{R}^0} \ell^1.$$

In view of (1.14), it follows that

$$(3.8) \quad \|\mathcal{F} : L^1 \rightarrow \ell^\infty\| \leq \sqrt{2}, \quad \|\mathcal{F}^{-1} : \ell^1 \rightarrow L^\infty\| \leq \sqrt{2}.$$

Therefore, by (3.7) and (2.16) we obtain

$$(3.9) \quad \|D\| \leq 2\|\tilde{R}^0| \ell^1\| \cdot \|v\|_1 = 2A(z; 1) \cdot \|v\|_1.$$

By Lemma 29, (6.25) in Appendix, we have that

$$(3.10) \quad \|\tilde{R}^0| \ell^1\| = A(z; 1) \lesssim \frac{\log N}{N}, \quad z \in \Lambda_N.$$

In view of (3.9) and (3.10), there is N_* such that

$$(3.11) \quad \|D : \ell^1 \rightarrow \ell^1\| < 1/2, \quad N \geq N_*, \quad z \in \Lambda_N.$$

Now, by (3.6), (3.8)–(3.11) it follows for $m \geq 3$, $N \geq N_*$, $z \in \Lambda_N$, that

$$\|U(m)\| \leq 2\|\tilde{R}^0| \ell^1\| \cdot \|D\|^2 \cdot \|D\|^{m-3} \leq 8(1/2)^{m-3} \|v\|_1^2 A^3(z; 1),$$

which yields, in view of (3.5),

$$\|B_N : L^1 \rightarrow \mathcal{C}\| \leq \int_{\Lambda_N} \sum_{m=3}^{\infty} \|U(m)\| dy \leq 16\|v\|_1^2 \int_{\Lambda_N} A^3(z; 1) dy.$$

Hence, from Corollary 31 it follows that

$$(3.12) \quad \|B_N : L^1 \rightarrow \mathcal{C}\| \lesssim 1/N.$$

3. Next we need to analyze the operator T_N . As before, we may explain that $T_N = \frac{1}{2\pi i} \int_{\Lambda_N} U(2) dy$. However, $\int_{\Lambda_N} A^2(z; 1) dy = \infty$, see (6.34) in Appendix, so – contrary to the case in Section 2 – we cannot integrate the estimate $\|U(2)\| \leq CA^2(z; 1)$ over Λ_N and get an estimate of $\|T_N\|$.

We go around this bump by *integrating first* in (1.35) and then analyzing the resulting operator by using its matrix representation with respect to the basis of eigenfunctions of the free operator L_{bc}^0 . Let

$$(3.13) \quad f = \sum f_k u_k \in L^1$$

so if $bc = Per^+$ or Per^-

$$(3.14) \quad R^0 V R^0 f = \sum_{m \in \Gamma_{bc}} \frac{1}{z - m^2} \left(\sum_{k \in \Gamma_{bc}} \frac{V(m - k) f(k)}{z - k^2} \right) u_m.$$

Our goal is to get the norm estimates, and if our results depend only on the norms it is sufficient to check estimates on dense subsets in L^1 , both for f and for v . Therefore, one may assume that all sums are finite.

Notice that

$$(3.15) \quad \frac{1}{2\pi i} \int_{\partial \Pi_N} \frac{dz}{(z - m^2)(z - k^2)} = \begin{cases} 0 & \text{if } |k|, |m| \leq N \text{ or } |k|, |m| > N, \\ \frac{1}{m^2 - k^2} & \text{if } |m| \leq N, |k| > N, \\ \frac{-1}{m^2 - k^2} & \text{if } |m| > N, |k| \leq N. \end{cases}$$

Therefore, the following holds.

Lemma 8. *For $bc = Per^\pm$, the operator T_N from (1.35) has a matrix representation*

$$(3.16) \quad T_N(m, k) = \begin{cases} -\frac{V(m-k)}{|m^2 - k^2|}, & (m, k) \in X(N), \\ 0, & (m, k) \notin X(N), \end{cases}$$

respectively, about the o.n.b. $\{u_m : m \in \Gamma_{Per^\pm}\}$ of eigenfunctions of the free operator $L_{Per^\pm}^0$, where

$$(3.17) \quad X(N) = \{(m, k) : m, k \in \Gamma_{Per^\pm}; |m| \leq N, |k| > N \text{ or } |m| > N, |k| \leq N\}.$$

We will use this matrix representation many times in what follows. Now, in view of (3.13) and (1.14), we have

$$(3.18) \quad \|Tf\|_\infty = \left\| \sum_{(m,k) \in X} \frac{-V(m-k)f(k)}{|m^2 - k^2|} u_m(x) \right\|_\infty \leq \sqrt{2} \|f\|_1 \sum_{(m,k) \in X} \frac{|V(m-k)|}{|m^2 - k^2|}.$$

By Lemma 32, Appendix,

$$(3.19) \quad \sum_{(m,k) \in X(N)} \frac{1}{|m^2 - k^2|} \leq 8 \cdot \frac{\pi^2}{8} < 10$$

for any N , and by Lemma 33, Appendix,

$$(3.20) \quad \sum_{\substack{(m,k) \in X(N) \\ |m-k| \leq H}} \frac{1}{|m^2 - k^2|} \leq 4 \cdot \frac{H}{N}.$$

Therefore,

$$(3.21) \quad \sum_{(m,k) \in X(N)} \frac{|V(m-k)|}{|m^2 - k^2|} = \sum_{\substack{(m,k) \in X(N) \\ |m-k| > H}} + \sum_{\substack{(m,k) \in X(N) \\ |m-k| \leq H}} \\ \leq 10v^*(H) + \|v\|_1 \cdot 4\frac{H}{N}.$$

This completes the proof of (3.3) if $bc = Per^\pm$.

4. The Dirichlet bc is done in the same way but some adjustments should be mentioned. For singular potentials, the matrix representation of the multiplication operator V comes from the formulas (1.21), (1.22) and (1.30). Of course, in the classical case where

$$v(x) \in L^1, \quad V(0) = \frac{1}{\pi} \int_0^\pi v(x) dx = 0,$$

we have

$$v(x) = Q'(x) \quad \text{with} \quad Q(x) = \int_0^x v(t) dt.$$

Now one can easily see that (1.30) holds with $(\tilde{V}(k))$ being the cosine coefficients of $v(x)$, i.e.,

$$Vu_k = \sum_{m=1}^{\infty} V_{mk} u_m, \quad u_k = \sqrt{2} \sin kx,$$

where

$$V_{mk} = \frac{1}{\sqrt{2}} \left(\tilde{V}(|m-k|) - \tilde{V}(m+k) \right), \quad \tilde{V}(k) = \frac{1}{\pi} \int_0^\pi v(x) \sqrt{2} \cos kx dx.$$

If $f = \sum_{k=1}^{\infty} f(k) u_k$ it follows that

$$(3.22) \quad R^0 V R^0 f = \sum_{m \in \Gamma_{bc}} \frac{1}{z - m^2} \left(\sum_{k \in \Gamma_{bc}} \frac{V_{mk} f(k)}{z - k^2} \right) u_m.$$

By (3.15), after integration we obtain the following matrix representation of the operator T_N .

Lemma 9. *For $bc = Dir$, the operator T_N from (1.35) has a matrix representation*

$$(3.23) \quad T_N(m, k) = \begin{cases} \frac{\tilde{V}(m+k) - \tilde{V}(|m-k|)}{\sqrt{2}|m^2 - k^2|}, & (m, k) \in X(N), \\ 0, & (m, k) \notin X(N), \end{cases}$$

about the o.n.b. $\{\sqrt{2} \sin kx, k \in \mathbb{N}\}$, where

$$(3.24) \quad X(N) = \{(m, k) : m, k \in \mathbb{N} : k \leq N < m \text{ or } m \leq N < k\}.$$

With Formula (3.23) a proper adjustment in inequalities (3.18)–(3.21) leads to the estimate (3.3) in the case $bc = Dir$. \square

5. The case where the potential v is a derivative of a BV -function.

In the case of Dirichlet bc and a real-valued potential $v = Q'$, where Q is a π -periodic function of bounded variation on $[0, \pi]$, i.e.,
(3.25)

$$Var(Q, [0, \pi]) = \sup \left\{ \sum_{i=1}^n |Q(x_i) - Q(x_{i-1})| : 0 = x_0 < x_1 < \dots < x_n = \pi \right\} < \infty,$$

V. A. Vinokurov and V. A. Sadovnichii [55] showed that

$$(3.26) \quad \|(S_N - S_N^0)f\|_\infty \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad \forall f \in L^1([0, \pi]).$$

We consider $bc = Per^+$ or Per^- as well and drop the requirement for v to be real-valued. The following is true.

Theorem 10. *Let $v = Q'$, where Q is a complex-valued function of bounded variation on $[0, \pi]$. Then, for $bc = Per^\pm$ and $bc = Dir$, the equiconvergence (3.26) holds.*

Proof. Consider the diagram

$$(3.27) \quad \mathcal{C}^* \xrightarrow{\mathcal{F}} \ell^\infty \xrightarrow{\tilde{R}^0} \ell^1 \xrightarrow{J} \mathcal{C} \xrightarrow{V} \mathcal{C}^*,$$

where \mathcal{C}^* is the space of continuous linear functionals on $\mathcal{C} = C([0, \pi])$. If $v \in \mathcal{C}^*$ and $f \in \mathcal{C}$, then the product $v \cdot f$ is an element of \mathcal{C}^* such that

$$\langle v \cdot f, \varphi \rangle = \langle v, f \cdot \varphi \rangle \quad \forall \varphi \in \mathcal{C}.$$

As in the proof of Theorem 7 we come to the conclusion

$$(3.28) \quad \|S_N - S_N^0 : L^1 \rightarrow \mathcal{C}\| \leq M \left(v^*(H) + \|v\|_{\mathcal{C}^*} \frac{H}{N} \right).$$

If $v \in L^1$ then $v^*(H) \rightarrow 0$ as $H \rightarrow 0$. But for $v \in \mathcal{C}^*$ we can only say that $v^*(H) \leq M_1 Var(Q, [0, \pi])$ (see [56, Theorem II.4.12]).

With $H = N$, we obtain from (3.28)

$$(3.29) \quad \|S_N - S_N^0 : L^1 \rightarrow \mathcal{C}\| \leq M_{bc},$$

where M_{bc} is a constant which does not depend on N . However,

$$(3.30) \quad \|(S_N - S_N^0)\varphi\|_\infty \rightarrow 0$$

if φ is smooth enough, say $\varphi \in C^2([0, \pi])$, and the space $C^2([0, \pi])$ is dense in $L^1([0, \pi])$. This explains that (3.29) leads to (3.26). \square

6. The inequalities from Subsection 3.3 could be adjusted to analysis of "weakly singular" potentials $v \in H^{-\alpha}$, $0 < \alpha \leq 1/2$, and one may show for such potentials that $\|S_N - S_N^0 : L^1 \rightarrow L^\infty\| \rightarrow 0$ as $N \rightarrow \infty$. But we prefer to analyze these potentials in the next section, together with "strongly singular" potentials $v \in H^{-\alpha}$, $1/2 < \alpha < 1$.

4. THE CASE OF POTENTIALS $v \in H^{-\alpha}$, $0 < \alpha < 1$.

1. Here we study how the equiconvergence depends on the singularity of v (measured by the appropriate scale of Sobolev spaces).

Recall that if $\Omega = (\Omega(k))_{k \in \mathbb{Z}}$ is a sequence of positive numbers (weight sequence), one may consider the weighted sequence space

$$\ell^2(\Omega, 2\mathbb{Z}) = \left\{ x = (x_k) : \sum_{k \in 2\mathbb{Z}} (|x_k| \Omega(k))^2 < \infty \right\}$$

and the corresponding Sobolev space

$$(4.1) \quad H(\Omega) = \left\{ f = \sum_{k \in 2\mathbb{Z}} f_k e^{ikx} : (f_k) \in \ell^2(\Omega) \right\}.$$

In particular, consider the Sobolev weights

$$(4.2) \quad \Omega_\alpha(k) = (1 + k^2)^{\alpha/2}, \quad k \in \mathbb{Z}, \quad \alpha \in \mathbb{R},$$

and the logarithmic weights

$$(4.3) \quad \omega_\beta(k) = (\log(e + |k|))^\beta, \quad k \in \mathbb{Z}, \quad \beta \in \mathbb{R}.$$

Let H^α and h^β denote the corresponding Sobolev spaces (4.1). Of course, $H^\alpha \subset h^\beta$ if $\alpha > 0$ and $h^\beta \subset H^\alpha$ if $\alpha < 0$ for any β .

The following lemma will be useful.

Lemma 11. *Let $g \in C^1([0, \pi])$.*

- (a) *If $f \in H^\alpha$, $-1/2 < \alpha < 1/2$, then $f \cdot g \in H^\alpha$.*
- (b) *If $f \in h^\beta$, $-\infty < \beta < \infty$, then $f \cdot g \in h^\beta$.*

Proof is given in [9, Appendix].

Now we consider potentials $v \in H^{-\alpha}$, $0 < \alpha < 1$, i.e. $v \in H_{per}^{-1}$ and

$$(4.4) \quad v = \sum_{k \in 2\mathbb{Z}} V(k) e^{ikx}, \quad V(0) = 0, \quad \sum_k \frac{|V(k)|^2}{(1 + k^2)^\alpha} < \infty.$$

or equivalently (see (1.17)), $v = Q'$ and

$$(4.5) \quad Q = \sum_{k \in 2\mathbb{Z}} q(k) e^{ikx}, \quad V(k) = ikq(k), \quad \sum_k |q(k)|^2 (1 + k^2)^{1-\alpha} < \infty.$$

Notice that $v \in H^{-\alpha}$ if and only if $Q \in H^{1-\alpha}$.

In the context of Dirichlet boundary conditions, we may consider the spaces $\tilde{H}^{-\alpha}$, $0 < \alpha < 1/2$ or $1/2 < \alpha < 1$, of all potentials $v \in H_{per}^{-1}$ such that

$$(4.6) \quad v = \sum_{m=0}^{\infty} \tilde{V}(m) \sqrt{2} \cos mx, \quad \tilde{V}(0) = 0, \quad \sum_m \frac{|\tilde{V}(m)|^2}{(1 + m^2)^\alpha} < \infty,$$

or equivalently (see Formula (1.22)), $v = Q'$ for some Q such that

$$(4.7) \quad Q = \sum_{m \in \mathbb{N}} \tilde{q}(m) \sqrt{2} \sin mx, \quad \tilde{V}(m) = m \tilde{q}(m), \quad \sum_m |\tilde{q}(m)|^2 m^{2(1-\alpha)} < \infty.$$

It turns out that for $0 < \alpha < 1/2$ the choice of an additive constant for Q (see Remark 2 and (1.23)) is essential. Indeed, then (4.7) and the Cauchy inequality imply

$$\sum_{m=1}^{\infty} |\tilde{q}(m)| \leq \left(\sum_m |\tilde{q}(m)|^2 m^{2(1-\alpha)} \right)^{1/2} \left(\sum_m m^{2(\alpha-1)} \right)^{1/2} < \infty, \quad 0 < \alpha < \frac{1}{2}.$$

Therefore, if (4.7) holds with $\alpha \in (0, 1/2)$, then the function $Q(x)$ is continuous, and $Q(0) = 0$.

Proposition 12. *If $0 < \alpha < 1/2$ or $1/2 < \alpha < 1$, then $\tilde{H}^{-\alpha} = H^{-\alpha}$.*

Proof. It is known that the discrete Hilbert transform

$$(4.8) \quad \mathcal{H} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z}), \quad (\mathcal{H}x)_n = \sum_{k \neq n} \frac{x_k}{n-k}$$

act continuously in the weighted spaces $\ell^2(\mathbb{Z}, \Omega_\delta)$, $\Omega_\delta(k) = (1 + k^2)^{\delta/2}$ if $|\delta| < 1/2$ ([19, Theorem 10], see also Lemma 32 in [9, Appendix]). We use this fact several times in the proof.

First we show that $\tilde{H}^{-\alpha} \subset H^{-\alpha}$. Suppose $v \in \tilde{H}^{-\alpha}$; then (4.7) holds, so

$$(4.9) \quad (\tilde{q}(m))_{m \in \mathbb{N}} \in \ell^2(m^{1-\alpha}, \mathbb{N}).$$

Taking into account that

$$(4.10) \quad \int_0^\pi \sin mx e^{-i2kx} dx = \begin{cases} 0 & m = 2s > 0, |k| \neq s; \\ \frac{\pi}{2i} \operatorname{sgn}(k) & m = 2|k|; \\ \frac{1}{2s-1-2k} + \frac{1}{2s-1+2k} & m = 2s-1, \end{cases}$$

we evaluate $q(2k) = \frac{1}{\pi} \int_0^\pi (\sum_{m=1}^\infty \tilde{q}(m) \sqrt{2} \sin mx) e^{-i2kx} dx$:

$$(4.11) \quad q(2k) = \frac{-i}{\sqrt{2}} \tilde{q}(2|k|) \operatorname{sgn}(k) + \frac{\sqrt{2}}{\pi} \sum_{s=1}^\infty \tilde{q}(2s-1) \left(\frac{1}{2s-1-2k} + \frac{1}{2s-1+2k} \right).$$

In the case $1/2 < \alpha < 1$, the latter sum can be regarded as a discrete Hilbert transform of the sequence $\xi = (\xi_k)_{k \in \mathbb{Z}}$, where

$$\xi_k = 0 \quad \text{if } k \text{ is even}, \quad \xi_k = -\operatorname{sgn}(k) \tilde{q}(|k|) \quad \text{if } k \text{ is odd},$$

that is, we have

$$q(2k) = \frac{-i}{\sqrt{2}} \tilde{q}(2|k|) \operatorname{sgn}(k) + \frac{\sqrt{2}}{\pi} (\mathcal{H}\xi)_{2k}, \quad (\mathcal{H}\xi)_n = \sum_{k \neq n} \frac{\xi_k}{n-k}.$$

Moreover, by (4.9) we have $\xi \in \ell^2(\Omega_{(1-\alpha)}, \mathbb{Z})$ with $0 < 1 - \alpha < 1/2$, so it follows that $\{(H\xi)_{2k}\} \in \ell^2(\Omega_{(1-\alpha)}, 2\mathbb{Z})$. Therefore, (4.5) holds, i.e., $v \in H^{-\alpha}$. Hence $\tilde{H}^{-\alpha} \subset H^{-\alpha}$ if $1/2 < \alpha < 1$.

In the case $0 < \alpha < 1/2$, we multiply (4.11) by $(2k)i$ and obtain (4.12)

$$V(2k) = \frac{1}{\sqrt{2}}\tilde{V}(2|k|) + i\frac{\sqrt{2}}{\pi} \sum_{s=1}^{\infty} \tilde{V}(2s-1) \left(\frac{1}{2s-1-2k} - \frac{1}{2s-1+2k} \right)$$

because

$$2k \left(\frac{1}{2s-1-2k} + \frac{1}{2s-1+2k} \right) = (2s-1) \left(\frac{1}{2s-1-2k} - \frac{1}{2s-1+2k} \right).$$

The sum in (4.12) may be considered as a discrete Hilbert transform of the sequence $u = u_k$, where

$$u_k = 0 \quad \text{if } k \text{ is even,} \quad u_k = -\tilde{V}(|k|) \quad \text{if } k \text{ is odd,}$$

that is, we have

$$V(2k) = \frac{1}{\sqrt{2}}\tilde{V}(2|k|) \operatorname{sgn}(k) + \frac{i\sqrt{2}}{\pi}(\mathcal{H}u)_{2k}.$$

Since $(\tilde{V}(m)) \in \ell^2(\Omega_{-\alpha})$, $0 < \alpha < 1/2$, we obtain

$$u \in \ell^2(\Omega_{-\alpha}) \Rightarrow \mathcal{H}u \in \ell^2(\Omega_{-\alpha}) \Rightarrow (V(2k)) \in \ell^2(\Omega_{-\alpha}) \Rightarrow v \in H^{-\alpha}.$$

This completes the proof of the inclusion $\tilde{H}^{-\alpha} \subset H^{-\alpha}$.

Next we show that $\tilde{H}^{-\alpha} \supset H^{-\alpha}$. Let $v \in H^{-\alpha}$, $1/2 < \alpha < 1$. Since $\tilde{q}(m) = \frac{1}{\pi} \int_0^\pi Q(x) \sqrt{2} \sin mx \, dx$, we obtain

$$\tilde{q}(m) = \begin{cases} \frac{i}{\sqrt{2}}[(q(2k) - q(-2k))] & \text{if } m = 2k, \\ \frac{i}{\sqrt{2}}[q_1(2k+2) - q_1(-2k)] & \text{if } m = 2k+1, \end{cases}$$

where $\{q_1(s), s \in 2\mathbb{Z}\}$ are the Fourier coefficients of the function $Q_1(x) = e^{ix} \cdot Q(x)$. By Lemma 11, we have

$$Q \in H^{1-\alpha} \iff Q_1 \in H^{1-\alpha} \quad \text{if } 1/2 < \alpha < 1.$$

Therefore, if $\frac{1}{2} < \alpha < 1$ and $v \in H^{-\alpha}$, then we obtain

$$v \in H^{-\alpha} \Rightarrow (4.5) \Rightarrow Q \in H^{1-\alpha} \Rightarrow Q_1 \in H^{1-\alpha} \Rightarrow (4.7) \Rightarrow (4.6) \Rightarrow v \in \tilde{H}^{-\alpha},$$

i.e., $\tilde{H}^{-\alpha} \supset H^{-\alpha}$ if $\frac{1}{2} < \alpha < 1$.

Next we show that $\tilde{H}^{-\alpha} \supset H^{-\alpha}$ in the case $0 < \alpha < \frac{1}{2}$. Let $v \in H^{-\alpha}$. Then, by (4.5) and the Cauchy inequality, $\sum |q(2k)| < \infty$, so $Q(x) = \sum q(2k)e^{i2kx}$ is continuous function and we have

$$(4.13) \quad Q(0) = \sum_{\mathbb{Z}} q(2k) = 0 \Rightarrow q(0) = -\sum_{k \neq 0} q(2k).$$

We evaluate the coefficients $\tilde{q}(m)$, $m \in \mathbb{N}$:

$$\tilde{q}(m) = \frac{1}{\pi} \sqrt{2} \int_0^\pi Q(x) \sin mx \, dx = \sum_{k \in \mathbb{Z}} q(2k) \frac{\sqrt{2}}{\pi} \int_0^\pi e^{i2kx} \sin mx \, dx.$$

In view of (4.10), we obtain

$$(4.14) \quad \tilde{q}(m) = \frac{i}{\sqrt{2}} [q(m) - q(-m)] \quad \text{for even } m,$$

$$(4.15) \quad \tilde{q}(m) = \frac{\sqrt{2}}{\pi} \sum_{k \in \mathbb{Z}} q(2k) \left(\frac{1}{m+2k} + \frac{1}{m-2k} \right) \quad \text{for odd } m.$$

By (4.13), (4.15) implies

$$(4.16) \quad \tilde{q}(m) = \frac{\sqrt{2}}{\pi} \sum_{k \neq 0} q(2k) \left(\frac{1}{m+2k} + \frac{1}{m-2k} - \frac{2}{m} \right) \quad \text{for odd } m.$$

Since $\tilde{V}(m) = m \tilde{q}(m)$, $V(2k) = i(2k)q(2k)$ and

$$\frac{1}{m+2k} + \frac{1}{m-2k} - \frac{2}{m} = \frac{2k}{m} \left(\frac{1}{m-2k} - \frac{1}{m+2k} \right),$$

from (4.16) it follows that

$$(4.17) \quad \tilde{V}(m) = \frac{1}{i} \sum_{k \neq 0} \frac{V(2k)}{m-2k} - \frac{1}{i} \sum_{k \neq 0} \frac{V(2k)}{m+2k} = \frac{1}{i} [(\mathcal{H}w)_m - (\mathcal{H}w)_{-m}] \quad \text{for odd } m,$$

where

$$w_s = \begin{cases} 0 & \text{if } s = 2k-1, \\ V(2k) & \text{if } s = 2k. \end{cases}$$

By (4.4), we know that $w \in \ell^2(\Omega_{-\alpha}, \mathbb{Z})$, so $\mathcal{H}w \in \ell^2(\Omega_{-\alpha}, \mathbb{Z})$ also. Therefore, by (4.14) and (4.17) we conclude that $(\tilde{V}(m)) \in \ell^2(\Omega_{-\alpha}, \mathbb{Z})$, i.e., $v \in \tilde{H}^{-\alpha}$. Hence, $\tilde{H}^{-\alpha} \supset H^{-\alpha}$ if $0 < \alpha < 1/2$. This completes the proof. \square

Remark 13. The definition (4.6) of the classes $\tilde{H}^{-\alpha}$ for $1/2 < \alpha < 1$ is correct (although if we add a constant C to Q then for odd m the coefficients $\tilde{V}(m)$ will change by $2C$). But we cannot define a class $\tilde{H}^{-1/2}$ by (4.6) with $\alpha = 1/2$ because such a definition will depend essentially on the choice of an arbitrary additive constant.

2. Our main result in this section is the following theorem.

Theorem 14. Let S_N, S_N^0 be the spectral projections defined by (1.33) for the Hill operators $L_{bc}(v)$ and L_{bc}^0 subject to the boundary conditions $bc = \text{Per}^\pm$ or Dir .

(a) If $v \in H^{-\alpha}$ with $\alpha \in (0, 1/2)$, then

$$(4.18) \quad \|S_N - S_N^0 : L^1 \rightarrow L^\infty\| = o(N^{\alpha-\frac{1}{2}}), \quad N \rightarrow \infty.$$

(b) If $bc = \text{Per}^\pm$ and $v \in H^{-1/2}$ or $bc = \text{Dir}$ and $v = Q'$ with $Q = \sum_{m \in \mathbb{N}} \tilde{q}(m) \sin mx$, $\sum_{m \in \mathbb{N}} |\tilde{q}(m)|^2 m < \infty$, then

$$(4.19) \quad \|S_N - S_N^0 : L^1 \rightarrow L^\infty\| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

(c) If $v \in H^{-\alpha}$ with $\alpha \in (1/2, 1)$ and $a = \frac{2}{3-2\alpha}$ (i.e., $\frac{1}{a} = \frac{3}{2} - \alpha$), then

$$(4.20) \quad \|S_N - S_N^0 : L^a \rightarrow L^\infty\| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Proof. By (1.34), we have $S_N - S_N^0 = T_N + B_N$, where the operators T_N and B_N are defined by (1.35) and (1.36). We estimate appropriate norms of the operators T_N and B_N in Propositions 17 and 19 below. The results obtained there are of independent interest – they are more general than the estimates leading to (4.18)–(4.20).

First we prove (a). Let $v \in H^{-\alpha}$ with $\alpha \in (0, 1/2)$. Then Proposition 12 and (4.4)–(4.7) imply that

$$(4.21) \quad q \in \ell^2(\Omega), \quad \tilde{q} \in \ell^2(\Omega), \quad \Omega(k) = (1 + k^2)^{(1-\alpha)/2}.$$

Therefore, by (b) in Proposition 19 (with $a = 1$ and $\delta = 1 - \alpha$ in (4.68)) we have

$$\|T_N : L^1 \rightarrow L^\infty\| \lesssim \frac{H}{N^{1/2}} + \mathcal{E}_{H_N}^\Omega(q) N^{\alpha-\frac{1}{2}}.$$

So, choosing $H_N = N^\alpha / \log N$ we obtain

$$\|T_N : L^1 \rightarrow L^\infty\| = o\left(N^{\alpha-\frac{1}{2}}\right)$$

because $\mathcal{E}_{H_N}^\Omega(q) \rightarrow 0$ as $N \rightarrow \infty$.

On the other hand, by (c) in Proposition 17 (see (4.46) with $\delta = 1 - \alpha$) we have

$$\|B_N : L^1 \rightarrow L^\infty\| \lesssim N^{\alpha-1} (\log N)^2 = o\left(N^{\alpha-\frac{1}{2}}\right).$$

Hence (4.18) holds.

By the assumption of (b), it follows that (4.21) holds with $\alpha = 1/2$. Indeed, if $v \in H^{-1/2}$, this follows from (4.4) and (4.5), and it is assumed that $\tilde{q} \in \ell^2(\Omega)$ with $\Omega(k) = (1 + k^2)^{1/4}$. Now the same argument as in the proof of (a) shows that (4.19) holds.

Finally, we prove (c). Let $v \in H^{-\alpha}$ with $\alpha \in (1/2, 1)$. As in the proof of (a), Proposition 12 and (4.4)–(4.7) imply that (4.21) holds. Therefore, by (b) in Proposition 19 with $a = \frac{2}{3-2\alpha}$, $\delta = 1 - \alpha$ and $H = N^{a/4}$ in (4.68), we obtain

$$\|T_N : L^a \rightarrow L^\infty\| \lesssim \frac{1}{N^{1/4}} + \mathcal{E}_{N^{a/4}}^\Omega(q) \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

On the other hand, in view (c) in Proposition 17 (see (4.46) with $\delta = 1 - \alpha$) we have

$$\|B_N : L^a \rightarrow L^\infty\| \leq \|B_N : L^1 \rightarrow L^\infty\| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Hence (4.20) holds, which completes the proof. \square

3. Our proofs are based on the Fourier analysis approach to the theory of Hill operators with singular potentials developed in [6]. Below we recall some basic formulas related to this approach.

In general, there are no good estimates for the norms of $R_\lambda^0 V$ and $V R_\lambda^0$ in the case of singular potentials. Therefore, now we write the standard perturbation type formula for the resolvent R_λ in the form

$$(4.22) \quad R_\lambda = R_\lambda^0 + R_\lambda^0 V R_\lambda^0 + \cdots = K_\lambda^2 + \sum_{m=1}^{\infty} K_\lambda (K_\lambda V K_\lambda)^m K_\lambda,$$

where

$$(4.23) \quad (K_\lambda)^2 = R_\lambda^0.$$

We define an operator $K = K_\lambda$ with the property (4.23) by its matrix representation

$$(4.24) \quad K_{jm} = \frac{1}{(\lambda - j^2)^{1/2}} \delta_{jm}, \quad j, m \in \Gamma_{bc},$$

where

$$z^{1/2} = \sqrt{r} e^{i\varphi/2} \quad \text{if} \quad z = r e^{i\varphi}, \quad 0 \leq \varphi < 2\pi.$$

Then R_λ is well-defined if

$$(4.25) \quad \|K_\lambda V K_\lambda : \ell^2(\Gamma_{bc}) \rightarrow \ell^2(\Gamma_{bc})\| < 1.$$

In view of (1.29) and (4.24), the matrix representation of KVK for periodic or anti-periodic boundary conditions $bc = Per^\pm$ is

$$(4.26) \quad (KVK)_{jm} = \frac{V(j-m)}{(\lambda - j^2)^{1/2}(\lambda - m^2)^{1/2}} = i \frac{(j-m)q(j-m)}{(\lambda - j^2)^{1/2}(\lambda - m^2)^{1/2}},$$

where $j, m \in 2\mathbb{Z}$ for $bc = Per^+$, and $j, m \in 1 + 2\mathbb{Z}$ for $bc = Per^-$. Therefore, we have for its Hilbert-Schmidt norm (which dominates its ℓ^2 -norm)

$$(4.27) \quad \|KVK\|_{HS}^2 = \sum_{j,m \in \Gamma_{Per^\pm}} \frac{(j-m)^2 |q(j-m)|^2}{|\lambda - j^2| |\lambda - m^2|}.$$

In the case $bc = Dir$, we obtain by (1.30) and (4.24) that

$$(4.28) \quad (KVK)_{jm} = \frac{1}{\sqrt{2}} \frac{|j-m| \tilde{q}(|j-m|) - (j+m) \tilde{q}(j+m)}{(\lambda - j^2)^{1/2}(\lambda - m^2)^{1/2}}, \quad j, m \in \mathbb{N}.$$

Thus,

$$(4.29) \quad \|KVK\|_{HS}^2 \leq \sum_{j,m \in \mathbb{N}} \frac{(j-m)^2 |\tilde{q}(j-m)|^2 + (j+m)^2 |\tilde{q}(j+m)|^2}{|\lambda - j^2| |\lambda - m^2|}.$$

In view of (4.27) and (4.29), we can estimate from above the Hilbert-Schmidt norm $\|KVK\|_{HS}$ by one and the same formula in all three cases

$bc = Per^+, Per^-, Dir$. Indeed, if we set $q(k) = 0$ for $k \in 2\mathbb{Z}+1$ if $bc = Per^\pm$, and $q(k) = \tilde{q}(|k|)$ if $bc = Dir$, then $q \in \ell^2(\mathbb{Z})$ and we have

$$(4.30) \quad \|KVK\|_{HS}^2 \leq \sum_{j,m \in \mathbb{Z}} \frac{(j-m)^2 |q(j-m)|^2}{|\lambda - j^2| |\lambda - m^2|}, \quad bc = Per^\pm, Dir.$$

4. Next we estimate the Hilbert-Schmidt norm of the operator $K_\lambda V K_\lambda$ for $\lambda = N^2 + N + iy$, $y \in \mathbb{R}$. For a sequence $q = (q(k)) \in \ell^2$, or $q = (q(k)) \in \ell^2(\Omega)$ we set

$$(4.31) \quad \mathcal{E}_M(q) = \left(\sum_{|k| \geq M} |q(k)|^2 \right)^{1/2}, \quad \mathcal{E}_M^\Omega(q) = \left(\sum_{|k| \geq M} |q(k)|^2 (\Omega(k))^2 \right)^{1/2}.$$

Lemma 15. For $q = (q(k))_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ we set

$$(4.32) \quad \psi_N(y) = \sum_{j,m \in \mathbb{Z}} \frac{(j-m)^2 |q(j-m)|^2}{|\lambda - j^2| |\lambda - m^2|}, \quad \lambda = N^2 + N + iy.$$

Then

$$(4.33) \quad \psi_N(y) \leq N^2 \left(\frac{\|q\|^2}{N} + 16(\mathcal{E}_{\sqrt{N}}(q))^2 \right) b_N(y) + 16(\mathcal{E}_{4N}(q))^2 a_N(y),$$

where

$$(4.34) \quad a_N(y) = \sum_{k \in \mathbb{Z}} \frac{1}{|\lambda - k^2|}, \quad b_N(y) = \sum_{k \in \mathbb{Z}} \frac{1}{|\lambda - k^2|^2}, \quad \lambda = N^2 + N + iy.$$

Moreover, if $|y| \geq N^8$, then we have

$$(4.35) \quad \psi_N(y) \leq N^2 \left(\frac{\|q\|^2}{N} + 16(\mathcal{E}_{\sqrt{N}}(q))^2 \right) b_N(y) + \left(\frac{\|q\|^2}{\sqrt{|y|}} + 16(\mathcal{E}_{|y|^{1/4}}(q))^2 \right) a_N(y).$$

Proof. In view of (4.32),

$$(4.36) \quad \psi_N(y) = \sum_{s \in \mathbb{Z}} \left(\sum_{m \in \mathbb{Z}} \frac{s^2 |q(s)|^2}{|\lambda - (m+s)^2| |\lambda - m^2|} \right) = \sigma_1 + \sigma_2 + \sigma_3,$$

where

$$(4.37) \quad \sigma_1 = \sum_{|s| \leq \sqrt{N}} \dots, \quad \sigma_2 = \sum_{\sqrt{N} < |s| \leq 4N} \dots, \quad \sigma_3 = \sum_{|s| > 4N} \dots.$$

The Cauchy inequality implies that

$$(4.38) \quad \sum_{m \in \mathbb{Z}} \frac{1}{|\lambda - m^2| |\lambda - (m+s)^2|} \leq \sum_{m \in \mathbb{Z}} \frac{1}{|\lambda - m^2|^2}.$$

Therefore, in view of (4.34), it follows that

$$(4.39) \quad \sigma_1 \leq \left(\sum_{|s| \leq \sqrt{N}} s^2 |q(s)|^2 \right) b_N(y) \leq N \|q\|^2 b_N(y)$$

and

$$(4.40) \quad \sigma_2 \leq \left(\sum_{\sqrt{N} < |s| \leq 4N} s^2 |q(s)|^2 \right) b_N(y) \leq (4N)^2 (\mathcal{E}_{\sqrt{N}}(q))^2 b_N(y).$$

Next we estimate σ_3 . In view of (4.36) and (4.37),

$$\sigma_3 \leq (\mathcal{E}_{4N}(q))^2 \cdot \sup_{|s| > 4N} \sum_m \frac{s^2}{|\lambda - (m+s)^2| |\lambda - m^2|}.$$

If $|s| > 4N$ and $\lambda = N^2 + N + iy$, then

$$(4.41) \quad \frac{s^2}{|\lambda - (m+s)^2| |\lambda - m^2|} \leq \frac{8}{|\lambda - m^2|} + \frac{8}{|\lambda - (m+s)^2|}.$$

Indeed, if $|m| \geq |s|/2$, then (since $|s|/4 > N$)

$$|\lambda - m^2| \geq m^2 - |\operatorname{Re} \lambda| \geq s^2/4 - (N^2 + N) > s^2/4 - 2(|s|/4)^2 \geq s^2/8,$$

so (4.41) holds. If $|m| < |s|/2$, then $|m+s| \geq |s|/2$, and as above it follows that $|\lambda - (m+s)^2| \geq s^2/8$, so (4.41) holds also. Therefore,

$$\sup_{|s| > 4N} \sum_m \frac{s^2}{|\lambda - (m+s)^2| |\lambda - m^2|} \leq \sum_m \frac{16}{|\lambda - m^2|} = 16 a_N(y),$$

so we obtain

$$(4.42) \quad \sigma_3 \leq 16 (\mathcal{E}_{4N}(q))^2 a_N(y).$$

Now, in view of (4.36), the estimates (4.39), (4.40) and (4.42) imply (4.33).

Next we prove (4.35). To this end we estimate $\sigma_3 = \sigma_3(y)$ for $|y| > N^8$. Then

$$\sigma_3 = \sum_{4N < |s| \leq |y|^{1/4}} \cdots + \sum_{|s| > |y|^{1/4}} \cdots = \sigma_{3,1} + \sigma_{3,2}.$$

If $|s| < |y|^{1/4}$, then

$$\frac{s^2}{|\lambda - (m+s)^2|} \leq \frac{|y|^{1/2}}{|Im \lambda|} = \frac{1}{|y|^{1/2}},$$

so

$$\sigma_{3,1} \leq \frac{1}{|y|^{1/2}} \left(\sum_{4N < |s| \leq |y|^{1/4}} |q(s)|^2 \right) \sum_m \frac{1}{|\lambda - m^2|} \leq \frac{\|q\|^2}{\sqrt{|y|}} a_N(y).$$

On the other hand, by (4.41)

$$\sigma_{3,2} \leq \left(\sum_{|s| > |y|^{1/4}} |q(s)|^2 \right) \cdot 16a_N(y) \leq 16(\mathcal{E}_{|y|^{1/4}}(q))^2 a_N(y),$$

which completes the proof. \square

Lemma 15 is a modification of [6, Lemma 19]. We need also the following lemma which is a modification of [6, Lemma 20].

Lemma 16. *In the above notations, for $bc = Per^\pm$ or Dir , if $h \geq N$ then* (4.43)

$$\sup\{\|K_\lambda V K_\lambda\|_{HS} : |Re \lambda| \leq N^2 + N, |Im \lambda| \geq h\} \lesssim \frac{(\log h)^{\frac{1}{2}}}{h^{1/4}} \|q\| + \mathcal{E}_{4\sqrt{h}}(q),$$

where q is replaced by \tilde{q} if $bc = Dir$.

We omit the proof because it is the same as the proof of Lemma 20 in [6].

5. We estimate the norm of the operator B_N by using Lemmas 15, 16 and Lemmas 26 and 29 from Appendix. Let $v = Q'$, and let $q = (q(2k))$ and $\tilde{q} = (\tilde{q}(m))$ be, respectively, the sequences of Fourier coefficients of Q about the o.n.b. $\{e^{i2kx}, k \in \mathbb{Z}\}$ and $\{\sqrt{2} \sin mx, m \in \mathbb{N}\}$.

Proposition 17. (a) *If $bc = Per^\pm$, then*

$$(4.44) \quad \|B_N\|_{L^1 \rightarrow L^\infty} \lesssim \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right) (\log N)^2 + \int_{N^2}^\infty \frac{1}{t} (\mathcal{E}_t(q))^2 dt.$$

If $bc = Dir$, then (4.44) holds with q replaced by \tilde{q} .

(b) *Suppose $\Omega(t)$, $t \in \mathbb{R}$, is a real function which is even, unbounded and increasing for $x > 0$. If $q \in \ell^2(\Omega)$ and $bc = Per^\pm$, then*

$$(4.45) \quad \|B_N\|_{L^1 \rightarrow L^\infty} \lesssim \left(\frac{\|q\|^2}{N} + \frac{\|q\|_{\ell^2(\Omega)}^2}{\Omega^2(\sqrt{N})} \right) (\log N)^2 + \int_{N^2}^\infty \frac{\|q\|_{\ell^2(\Omega)}^2}{t \Omega^2(t)} dt.$$

If $bc = Dir$ and $\tilde{q} \in \ell^2(\Omega)$, then (4.45) holds with q replaced by \tilde{q} .

(c) *If $bc = Per^\pm$ and $q \in \ell^2(\Omega)$ or $bc = Dir$ and $\tilde{q} \in \ell^2(\Omega)$, where $\Omega(k) = (1 + k^2)^{\delta/2}$, $\delta \in (0, 1)$, then*

$$(4.46) \quad \|B_N\|_{L^1 \rightarrow L^\infty} \lesssim N^{-\delta} (\log N)^2.$$

If $\Omega(k) = (\log(e + k))^\beta$, $\beta > 1$, and respectively, $bc = Per^\pm$ and $q \in \ell^2(\Omega)$, or $bc = Dir$ and $\tilde{q} \in \ell^2(\Omega)$, then

$$(4.47) \quad \|B_N\|_{L^1 \rightarrow L^\infty} \lesssim (\log N)^{2-2\beta} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Proof. Recall by (1.36) and (1.32) that

$$(4.48) \quad B_N = \frac{1}{2\pi i} \int_{\partial \Pi_N} \sum_{m=2}^{\infty} R_{\lambda}^0 (V R_{\lambda}^0)^m d\lambda,$$

where $\Pi_N = \{\lambda = x + iy : -\omega \leq x \leq N^2 + N, |y| \leq h\}$. In view of (4.23), $R_{\lambda}^0 (V R_{\lambda}^0)^m = K_{\lambda} (K_{\lambda} V K_{\lambda})^m K_{\lambda}$, so we have

$$(4.49) \quad \|R_{\lambda}^0 (V R_{\lambda}^0)^m\|_{L^1 \rightarrow L^{\infty}} \leq \|K_{\lambda}\|_{L^1 \rightarrow L^2} \|K_{\lambda} V K_{\lambda}\|_{L^2 \rightarrow L^2}^m \|K_{\lambda}\|_{L^2 \rightarrow L^{\infty}}.$$

By (4.24) and (6.2), it follows that

$$(4.50) \quad \|K_{\lambda}\|_{L^1 \rightarrow L^2}^2 = \|K_{\lambda}\|_{L^2 \rightarrow L^{\infty}}^2 = \sum_k \frac{1}{|\lambda - k^2|} = A(\lambda, 1).$$

Since the Hilbert-Schmidt norm dominates the L^2 -norm, by (4.49) and (4.50) the $\|\cdot\|_{L^1 \rightarrow L^{\infty}}$ norm of the integrand in (4.48) does not exceed

$$(4.51) \quad \sum_{m=2}^{\infty} \|R_{\lambda}^0 (V R_{\lambda}^0)^m\|_{L^1 \rightarrow L^{\infty}} \leq S(\lambda),$$

where

$$(4.52) \quad S(\lambda) := A(\lambda, 1) \sum_{m=2}^{\infty} \|K_{\lambda} V K_{\lambda}\|_{HS}^m.$$

The integral in (4.48) does not depend on the choice of the parameters $\omega > \omega_0$, $h > h_0$ in (1.32) because the integrand depends analytically on $\lambda = x + it$ if $x < -\omega_0$, $|t| > h_0$. Lemma 16 implies that if $|Im \lambda| = h$ then $\|K_{\lambda} V K_{\lambda}\|_{HS} \leq 1/2$ for large enough h . Therefore, in view of (4.51), (4.52) and Lemma 26 (Appendix, formula (6.11) with $r = 1$), if N is large enough then on the horizontal sides of the rectangle Π_N the norm of the integrand in (4.48) does not exceed

$$S(\lambda) \leq A(\lambda, 1) \lesssim h^{-1/2}, \quad |Im \lambda| = h \geq N^2.$$

Let Λ_N and Λ_N^- be the vertical lines

$$\Lambda_N = \{\lambda = N^2 + N + iy : y \in \mathbb{R}\}, \quad \Lambda_N^- = \{\lambda = -(N^2 + N) + iy : y \in \mathbb{R}\}.$$

Now, taking $\omega = N^2 + N$ and letting $h \rightarrow \infty$ we obtain (since the integrals on horizontal segments go to zero) that

$$(4.53) \quad B_N = \frac{1}{2\pi i} \int_{\Lambda_N} \sum_{m=2}^{\infty} R_{\lambda}^0 (V R_{\lambda}^0)^m d\lambda - \frac{1}{2\pi i} \int_{\Lambda_N^-} \sum_{m=2}^{\infty} R_{\lambda}^0 (V R_{\lambda}^0)^m d\lambda,$$

provided both integrals in (4.53) converge. Therefore, from (4.51) and (4.52) it follows that

$$\|B_N\|_{L^1 \rightarrow L^{\infty}} \leq \int_{\Lambda_N \cup \Lambda_N^-} S(\lambda) dy, \quad \lambda = \pm(N^2 + N) = iy.$$

In view of (4.27) or (4.29), one can easily see that $\int_{\Lambda_N^-} S(\lambda) dy \leq \int_{\Lambda_N} S(\lambda) dy$ because $S(-(N^2 + N) + iy) \leq S(N^2 + N + iy)$, which implies that

$$(4.54) \quad \|B_N\|_{L^1 \rightarrow L^\infty} \leq 2 \int_{\Lambda_N} S(\lambda) dy.$$

Moreover, for large enough N we have

$$(4.55) \quad \|K_\lambda V K_\lambda\|_{HS} < 1/2 \quad \text{for } \lambda \in \Lambda_N.$$

Indeed, in view of (4.30) and (4.33) in Lemma 15, we have

$$\|K_\lambda V K_\lambda\|_{HS} \leq \psi_N(y), \quad \lambda = N^2 + N + iy \in \Lambda_N,$$

with

$$\psi_N(y) \leq N^2 \left(\frac{\|q\|^2}{N} + 16(\mathcal{E}_{\sqrt{N}}(q))^2 \right) b_N(y) + 16(\mathcal{E}_{4N}(q))^2 a_N(y),$$

where $a_N(y)$ and $b_N(y)$ are given by (4.34). By Lemma 29, we have that

$$(4.56) \quad a_N(y) \lesssim \begin{cases} \frac{\log N}{N} & \text{if } |y| \leq N; \\ \frac{1}{N} \log(1 + \frac{N^2}{|y|}) & \text{if } N \leq |y| \leq N^2; \\ \frac{1}{\sqrt{|y|}} & \text{if } |y| \geq N^2; \end{cases}$$

$$(4.57) \quad b_N(y) \lesssim \begin{cases} \frac{1}{N^2} & \text{if } |y| \leq N; \\ \frac{1}{N|y|} & \text{if } N \leq |y| \leq N^2; \\ \frac{1}{|y|^{3/2}} & \text{if } |y| \geq N^2. \end{cases}$$

Since $\mathcal{E}_{\sqrt{N}}(q) \rightarrow 0$ as $N \rightarrow 0$, by (4.56) and (4.57) one can easily see that $\sup\{\psi_N(y) : y \in \mathbb{R}\} \rightarrow 0$ as $N \rightarrow 0$, which proves (4.55).

From (4.55) it follows that $\sum_{m=2}^\infty \|K_\lambda V K_\lambda\|_{HS}^m \leq \|K_\lambda V K_\lambda\|_{HS}^2$ if $\lambda \in \Lambda_N$. Thus, by (4.30), (4.32) and (4.34), we obtain

$$(4.58) \quad S(\lambda) \leq a_N(y) \psi_N(y) \quad \text{for } \lambda = N^2 + N + iy \in \Lambda_N.$$

In view of (4.54) and (4.58),

$$(4.59) \quad \|B_N\|_{L^1 \rightarrow L^\infty} \leq 2 \int_{\mathbb{R}} a_N(y) \psi_N(y) dy = 2(I_1 + I_2 + I_3 + I_4),$$

where

$$I_1 = \int_{|y| \leq N} \dots, \quad I_2 = \int_{N < |y| \leq N^2} \dots, \quad I_3 = \int_{N^2 < |y| \leq N^8} \dots, \quad I_4 = \int_{|y| > N^8} \dots.$$

Now we estimate I_1 . In view of (4.56) and (4.57), if $|y| \leq N$ then $a_N(y) \leq \frac{\log N}{N}$, and $b_N(y) \leq \frac{1}{N^2}$. Therefore, by (4.33),

$$\begin{aligned} a_N(y) \psi_N(y) &\lesssim \frac{\log N}{N} \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right) + (\mathcal{E}_{4N}(q))^2 \frac{(\log N)^2}{N^2} \\ &\lesssim \frac{\log N}{N} \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right). \end{aligned}$$

Therefore, we obtain

$$(4.60) \quad I_1 \leq 2N \cdot \max_{|y| \leq N} a_N(y) \psi_N(y) \lesssim \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right) \log N.$$

Next we estimate I_2 . In view of (4.56) and (4.57), if $N \leq |y| \leq N^2$ then $a_N(y) \leq \frac{1}{N} \log(1 + \frac{N^2}{|y|})$, and $b_N(y) \leq \frac{1}{N|y|}$. Therefore, (4.33) implies

$$\begin{aligned} a_N(y) \psi_N(y) &\lesssim \frac{1}{|y|} \log \left(1 + \frac{N^2}{|y|} \right) \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right) \\ &+ (\mathcal{E}_{4N}(q))^2 \left(\frac{1}{N} \log \left(1 + \frac{N^2}{|y|} \right) \right)^2 \lesssim \frac{1}{|y|} \log \left(1 + \frac{N^2}{|y|} \right) \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right) \end{aligned}$$

because $\frac{1}{|y|} \geq \frac{1}{N^2} \log \left(1 + \frac{N^2}{|y|} \right)$. Now, since

$$\int_N^{N^2} \frac{1}{y} \log(1 + N^2/y) dy \lesssim (\log N) \int_N^{N^2} \frac{1}{y} dy \lesssim (\log N)^2,$$

it follows that

$$(4.61) \quad I_2 \lesssim \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right) (\log N)^2.$$

If $N^2 \leq |y| \leq N^8$, then by (4.33), (4.56) and (4.57) it follows that

$$a_N(y) \psi_N(y) \lesssim \frac{N^2}{y^2} \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right) + \frac{1}{|y|} (\mathcal{E}_{4N}(q))^2.$$

So, taking into account that

$$\int_{N^2}^{N^8} \frac{1}{y^2} dy < \frac{1}{N^2}, \quad \int_{N^2}^{N^8} \frac{1}{y} dy = 6 \log N,$$

we obtain

$$(4.62) \quad I_3 \lesssim \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right) \log N.$$

To estimate I_4 , we use that the estimates (4.35), (4.56) and (4.57) imply, for $|y| > N^8$, that

$$a_N(y) \psi_N(y) \lesssim \frac{N^2}{y^2} \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right) + \frac{1}{|y|} \left(\frac{\|q\|^2}{\sqrt{y}} + (\mathcal{E}_{|y|^{1/4}}(q))^2 \right).$$

Since $\int_{N^8}^{\infty} \frac{1}{y^2} dy = \frac{1}{N^8}$ and $\int_{N^8}^{\infty} \frac{1}{y^{3/2}} dy = \frac{2}{N^4}$, it follows that

$$(4.63) \quad I_4 \lesssim \frac{\|q\|^2}{N^4} + \frac{(\mathcal{E}_{\sqrt{N}}(q))^2}{N^4} + \int_{N^8}^{\infty} \frac{1}{y} (\mathcal{E}_{|y|^{1/4}}(q))^2 dy.$$

In view of (4.59) and (4.60)–(4.63), we obtain that (4.44) holds, which completes the proof (a).

To prove (b), we use that

$$(\mathcal{E}_M(q))^2 = \sum_{|s| \geq M} |q(s)|^2 \leq \frac{1}{(\Omega(M))^2} \sum_{|s| \geq M} |q(s)|^2 (\Omega(s))^2 \leq \frac{\|q\|_\Omega^2}{(\Omega(M))^2},$$

so

$$(4.64) \quad \mathcal{E}_M(q) \leq \frac{1}{\Omega(M)} \mathcal{E}_M^\Omega(q) \leq \frac{\|q\|_\Omega}{\Omega(M)},$$

where

$$(4.65) \quad (\mathcal{E}_M^\Omega(q))^2 = \sum_{|s| \geq M} |q(s)|^2 (\Omega(s))^2.$$

Now (4.45) follows from (4.44) and (4.64), which proves (b). Finally, one can easily see that (c) follows from (b). \square

In the proofs of Propositions 17 and Proposition 25 in Section 5, we use Formula (4.53) (where B_N is written as a difference of two integrals over the lines Λ_N and Λ_N^-). This representation of B_N is good enough for our proofs.

However, in the context of L^1 -potentials (see Section 3, Formula (3.5)), it is explained (by using simple estimates from Appendix, Lemmas 26 and 27)) that the operator B_N equals only the integral over Λ_N . For singular potentials, it is more difficult to show that in Formula (4.53) the integral $\int_{\Lambda_N^-} \sum_{m=2}^\infty R_\lambda^0 (V R_\lambda^0)^m d\lambda = 0$, but it could be done by using estimates from the proofs of Propositions 15 and 25. More precisely, the following holds.

Remark 18. (a) If $v \in H^{-\alpha}$, $\alpha \in (0, 1)$, then

$$(4.66) \quad B_N = \frac{1}{2\pi i} \int_{\Lambda_N} \sum_{m=2}^\infty R_\lambda^0 (V R_\lambda^0)^m d\lambda, \quad N > N_*.$$

where the integral converges in the operator norm $\|\cdot\|_{L^1 \rightarrow L^\infty}$.

(b) If $v \in H^{-1}$ and we consider B_N as an operator from L^a to L^b , where $1 \leq a < 2 < b \leq \infty$ and $\frac{1}{a} - \frac{1}{b} < 1$, then (4.66) holds and the integral there converges in the operator norm $\|\cdot\|_{L^a \rightarrow L^b}$.

Proof. For potentials $v \in H^{-\alpha}$, $\alpha \in (0, 1)$, Formula (4.53) make sense because $\int_{\Lambda_N} S(\lambda) dy$ converge – see (4.51), (4.52), (4.54) and the estimates that follow. Using the same argument that leads to Formula (4.53) but with $\omega = M^2 + M$, $M \in \mathbb{N}$, $M \geq N$, we obtain

$$B_N = \frac{1}{2\pi i} \int_{\Lambda_N} \sum_{m=2}^\infty R_\lambda^0 (V R_\lambda^0)^m d\lambda - \frac{1}{2\pi i} \int_{\Lambda_M^-} \sum_{m=2}^\infty R_\lambda^0 (V R_\lambda^0)^m d\lambda$$

Therefore, in view of (4.51) and (4.52), we will prove (4.66) if we show that $\int_{\Lambda_M^-} S(\lambda) dy \rightarrow 0$ as $M \rightarrow \infty$. This follows from the Lebesgue Dominated Convergence Theorem, since by Lemma 27, Formula (6.15),

$$S(M^2 + M + iy) \lesssim A(M^2 + M + iy, 1) \lesssim (M^2 + y)^{-1/2} \rightarrow 0 \text{ as } M \rightarrow 0,$$

and $S(M^2 + M + iy) \leq g(y)$, where $g(y) = S(N^2 + N + iy)$ is integrable because $\int_{\Lambda_N} S(\lambda) dy < \infty$.

The proof of (b) is exactly the same, but it is based on inequalities from the proof of Proposition 25. Therefore, we omit the details. \square

6. Next we estimate the norms of the operator T_N .

Proposition 19. (a) If $bc = Per^\pm$, then

$$(4.67) \quad \|T_N\|_{L^2 \rightarrow L^\infty} \lesssim \|q\| \sqrt{H/N} + \mathcal{E}_H(q) + \mathcal{E}_N(q) (\log N)^{1/2}, \quad 0 < H < \frac{N}{2}.$$

If $bc = Dir$, then (4.67) holds with q replaced by \tilde{q} .

(b) If $bc = Per^\pm$ and $q \in \ell^2(\Omega)$, where $\Omega(k) = (1 + |k|^2)^{\delta/2}$ with $\delta \in (0, 1)$, and $1 \leq a < 2$, $a\delta < 1$, then

$$(4.68) \quad \|T_N\|_{L^a \rightarrow L^\infty} \lesssim \|q\| \frac{H^{\frac{1}{a}}}{N^{1/2}} + \mathcal{E}_H^\Omega(q) N^{\frac{1}{a} - \delta - 1/2}.$$

If $bc = Dir$ and $\tilde{q} \in \ell^2(\Omega)$, then (4.68) holds with q replaced by \tilde{q} .

(c) If $\Omega(k) = (\log(e + k))^\beta$, $\beta \geq 1/2$, $q \in \ell^2(\Omega)$ and $bc = Per^\pm$, then

$$(4.69) \quad \|T_N\|_{L^2 \rightarrow L^\infty} \lesssim \|q\| N^{-1/4} + (\mathcal{E}_{\sqrt{N}}^\Omega(q)) \cdot (\log N)^{1/2 - \beta}.$$

If $bc = Dir$ and $\tilde{q} \in \ell^2(\Omega)$, then (4.69) holds with q replaced by \tilde{q} .

Proof. Suppose $bc = Per^\pm$ and let (u_k) be, respectively, the canonical orthonormal basis (1.11) or (1.12). In view of (1.17), (1.29) and (1.35), if $f = \sum_k f_k u_k$ is the expansion of $f \in L^2([0, \pi])$, then (3.15) gives

$$\begin{aligned} T_N f &= \frac{1}{2\pi i} \int_{\partial \Pi_N} \sum_k f_k \sum_j i \frac{(j-k)q(j-k)}{(\lambda-j^2)(\lambda-k^2)} u_j(x) d\lambda = \\ &= \sum_{|k| \leq N} f_k \sum_{|j| > N} i \frac{(j-k)q(j-k)}{k^2 - j^2} u_j(x) + \sum_{|k| > N} f_k \sum_{|j| \leq N} i \frac{(j-k)q(j-k)}{j^2 - k^2} u_j(x) \end{aligned}$$

By (1.11) or (1.12), $|u_j(x)| \leq 1$. Therefore, we have

$$\|T_N(f)\|_\infty \leq \sum_{|k| \leq N} |f_k| \sum_{|j| > N} \frac{|q(j-k)|}{|j+k|} + \sum_{|k| > N} |f_k| \sum_{|j| \leq N} \frac{|q(j-k)|}{|j+k|}.$$

By the Hölder inequality, it follows that

$$\|T_N(f)\|_\infty \leq \|(f_k)\|_{\ell^{\bar{a}}} (\sigma_1(a, N))^{1/a} + \|(f_k)\|_{\ell^{\bar{a}}} (\sigma_2(a, N))^{1/a},$$

where $\frac{1}{a} + \frac{1}{\bar{a}} = 1$, $\|(f_k)\|_{\ell^{\bar{a}}} \leq 2\|f\|_{L^a}$ by the Young-Hausdorff theorem, and

$$\sigma_1(a, N) = \sum_{|k| \leq N} \left(\sum_{|j| > N} \frac{|q(j-k)|}{|j+k|} \right)^a, \quad \sigma_2(a, N) = \sum_{|k| > N} \left(\sum_{|j| \leq N} \frac{|q(j-k)|}{|j+k|} \right)^a.$$

Therefore,

$$(4.71) \quad \|T_N\|_{L^a \rightarrow L^\infty} \lesssim (\sigma_1(a, N) + (\sigma_2(a, N))^{1/a}.$$

The situation is similar if $bc = Dir$ and (u_k) is the corresponding canonical basis (1.13). If $f = \sum_{k \in \mathbb{N}} f_k u_k$ is the expansion of $f \in L^2([0, \pi])$, then by (1.22), (1.30), (1.35) and (3.15) we obtain

$$\begin{aligned} T_N f &= \frac{1}{2\pi i} \int_{\partial \Pi_N} \sum_k f_k \sum_j \frac{|j-k| \tilde{q}(|j-k|) - (j+k) \tilde{q}(j+k)}{\sqrt{2}(\lambda - j^2)(\lambda - k^2)} u_j(x) d\lambda = \\ &= \sum_{1 \leq k \leq N} f_k \sum_{j > N} \frac{\tilde{q}(j-k)}{j+k} \frac{1}{\sqrt{2}} u_j(x) + \sum_{1 \leq k \leq N} f_k \sum_{j > N} \frac{\tilde{q}(j+k)}{j-k} \frac{1}{\sqrt{2}} u_j(x) \\ &= \sum_{k > N} f_k \sum_{1 \leq j \leq N} \frac{\tilde{q}(k-j)}{j+k} \frac{1}{\sqrt{2}} u_j(x) + \sum_{k > N} f_k \sum_{1 \leq j \leq N} \frac{\tilde{q}(j+k)}{k-j} \frac{1}{\sqrt{2}} u_j(x). \end{aligned}$$

By (1.14), $|u_j(x)| \leq \sqrt{2}$, so using the Hölder inequality as above we obtain (4.71) holds with

$$(4.72) \quad \sigma_1(a, N) = \sum_{k=-N}^N \left(\sum_{j>N} \frac{|\tilde{q}(j-k)|}{j+k} \right)^a, \quad \sigma_2(a, N) = \sum_{k>N} \left(\sum_{j=-N}^N \frac{|\tilde{q}(k-j)|}{k+j} \right)^a.$$

In view of (4.70) and (4.72), if we set $q(k) = 0$ for $k \in 2\mathbb{Z} + 1$ in the case $bc = Per^\pm$, and $q(k) = \tilde{q}(|k|)$ in the case $bc = Dir$, and define σ_1, σ_2 by (4.70) with $j, k \in \mathbb{Z}$, then (4.71) holds in all three cases $bc = Per^\pm, Dir$. Next we estimate σ_1 and σ_2 in terms of remainders $\mathcal{E}_M(q)$.

Lemma 20. *If $q \in \ell^2(\mathbb{Z})$, then*

$$(4.73) \quad \sigma_1(a, N) \lesssim \frac{1}{N^{a/2}} \sum_{0 \leq k \leq N} (\mathcal{E}_{N+1-k}(q))^a + \begin{cases} (\mathcal{E}_N(q))^a N^{1-\frac{a}{2}}, & 1 \leq a < 2, \\ (\mathcal{E}_N(q))^2 \log N, & a = 2, \end{cases}$$

$$(4.74) \quad \sigma_2(a, N) \lesssim \sum_{k>N} (\mathcal{E}_{k-N}(q))^a \left(\frac{1}{k} - \frac{1}{N+k} \right)^{a/2} + \begin{cases} (\mathcal{E}_N(q))^a N^{1-\frac{a}{2}}, & 1 \leq a < 2, \\ (\mathcal{E}_N(q))^2 \log N, & a = 2. \end{cases}$$

Proof. Changing, if negative, k with $-k$ and j with $-j$ we obtain

$$\begin{aligned} \sigma_1(a, N) &\lesssim \sum_{0 \leq k \leq N} \left(\sum_{j>N} \frac{|q(j-k)|}{j+k} \right)^a + \sum_{0 \leq k \leq N} \left(\sum_{j>N} \frac{|q(-j+k)|}{j+k} \right)^a \\ &+ \sum_{0 \leq k \leq N} \left(\sum_{j>N} \frac{|q(j+k)|}{j-k} \right)^a + \sum_{0 \leq k \leq N} \left(\sum_{j>N} \frac{|q(-j-k)|}{j-k} \right)^a. \end{aligned}$$

By the Cauchy inequality, it follows that

$$\begin{aligned} \sigma_1(a, N) &\lesssim \sum_{0 \leq k \leq N} \left(\sum_{j > N} (|q(j-k)|^2 + |q(-j+k)|^2) \right)^{a/2} \left(\sum_{j > N} \frac{1}{(j+k)^2} \right)^{a/2} \\ &\quad + \sum_{0 \leq k \leq N} \left(\sum_{j > N} (|q(j+k)|^2 + |q(-j-k)|^2) \right)^{a/2} \left(\sum_{j > N} \frac{1}{(j-k)^2} \right)^{a/2}. \end{aligned}$$

Therefore,

$$\sigma_1(a, N) \lesssim \sigma_{1,1}(a, N) + \sigma_{1,2}(a, N),$$

where

$$\begin{aligned} \sigma_{1,1} &= \sum_{0 \leq k \leq N} (\mathcal{E}_{N+1-k}(q))^a \frac{1}{(N+k)^{a/2}} \lesssim \frac{1}{N^{a/2}} \sum_{0 \leq k \leq N} (\mathcal{E}_{N+1-k}(q))^a, \\ \sigma_{1,2} &= \sum_{0 \leq k \leq N} (\mathcal{E}_{N+1+k}(q))^a \frac{1}{(N+1-k)^{a/2}} \lesssim \begin{cases} (\mathcal{E}_N(q))^a N^{1-\frac{a}{2}}, & 1 \leq a < 2, \\ (\mathcal{E}_N(q))^2 \log N, & a = 2, \end{cases} \end{aligned}$$

because

$$\sum_{s > M} \frac{1}{s^2} \leq \frac{1}{M}, \quad \sum_{s=1}^N \frac{1}{s^{a/2}} \lesssim \int_1^N x^{-\frac{a}{2}} dx \lesssim \begin{cases} N^{1-\frac{a}{2}}, & 1 \leq a < 2, \\ \log N, & a = 2. \end{cases}$$

Thus, (4.73) holds.

Next we estimate $\sigma_2(a, N)$. As for $\sigma_1(a, N)$, we obtain

$$\begin{aligned} \sigma_2 &\lesssim \sum_{k > N} \left(\sum_{0 \leq j \leq N} (|q(j-k)|^2 + |q(-j+k)|^2) \right)^{a/2} \left(\sum_{0 \leq j \leq N} \frac{1}{(j+k)^2} \right)^{a/2} \\ &\quad + \sum_{k > N} \left(\sum_{0 \leq j \leq N} (|q(j+k)|^2 + |q(-j-k)|^2) \right)^{a/2} \left(\sum_{0 \leq j \leq N} \frac{1}{(k-j)^2} \right)^{a/2}. \end{aligned}$$

Therefore,

$$\sigma_2(a, N) \lesssim \sigma_{2,1}(a, N) + \sigma_{2,2}(a, N),$$

where

$$\begin{aligned} \sigma_{2,1} &= \sum_{k > N} (\mathcal{E}_{k-N}(q))^a \left(\frac{1}{k} - \frac{1}{N+k} \right)^{a/2}, \\ \sigma_{2,2} &= \sum_{k > N} (\mathcal{E}_k(q))^a \left(\frac{1}{k-N} - \frac{1}{k} \right)^{a/2} \lesssim \begin{cases} (\mathcal{E}_N(q))^a N^{1-a/2}, & 1 < a < 2, \\ (\mathcal{E}_N(q))^a \log N, & a = 2, \end{cases} \end{aligned}$$

because $\sum_{M_1}^{M_2} \frac{1}{s^2} \lesssim \frac{1}{M_1} - \frac{1}{M_2}$,

$$\sum_{k > N} \left(\frac{1}{k-N} - \frac{1}{k} \right)^{\frac{a}{2}} = N^{\frac{a}{2}} \left(\sum_{s=1}^N \frac{1}{s^{\frac{a}{2}}(N+s)^{\frac{a}{2}}} + \sum_{s=N+1}^{\infty} \frac{1}{s^{\frac{a}{2}}(N+s)^{\frac{a}{2}}} \right)$$

$$\lesssim N^{\frac{a}{2}} \left(\sum_{s=1}^N \frac{1}{s^{\frac{a}{2}} N^{\frac{a}{2}}} + \sum_{s=N+1}^{\infty} \frac{1}{s^a} \right) \lesssim N^{1-a/2} \quad \text{if } 1 < a < 2,$$

and

$$\sum_{k>N} \left(\frac{1}{k-N} - \frac{1}{k} \right) = \lim_{M \rightarrow \infty} \sum_1^M \left(\frac{1}{s} - \frac{1}{s+N} \right) = \sum_1^N \frac{1}{s} \lesssim \log N.$$

Hence, (4.74) follows. \square

The following lemma proves (4.67) and (4.69).

Lemma 21. *In the above notations, if $q \in \ell^2(\mathbb{Z})$ and $H \in (0, N/2)$, then*

$$(4.75) \quad \sigma_1(2, N) + \sigma_2(2, N) \lesssim \|q\|^2 \frac{H}{N} + (\mathcal{E}_H(q))^2 + (\mathcal{E}_N(q))^2 \cdot \log N.$$

Moreover, if $q \in \ell^2(\Omega, \mathbb{Z})$ with $\Omega(k) = (\log(e + |k|))^\beta$, $\beta > 1/2$, then

$$(4.76) \quad \sigma_1(2, N) + \sigma_2(2, N) \lesssim \|q\|^2 N^{-1/2} + (\mathcal{E}_{\sqrt{N}}^\Omega(q))^2 \cdot (\log N)^{1-2\beta}.$$

Proof. Indeed, (4.75) follows from (4.73) and (4.74) because

$$\sum_{k=0}^N (\mathcal{E}_{N+1-k}(q))^2 \leq \sum_{k=0}^{N-H} (\mathcal{E}_H(q))^2 + \sum_{N-H}^N \|q\|^2 \leq N \cdot (\mathcal{E}_H(q))^2 + H \|q\|^2,$$

and

$$\begin{aligned} \sum_{k>N} (\mathcal{E}_{k-N}(q))^2 \left(\frac{1}{k} - \frac{1}{k+N} \right) &\leq \sum_{k=N+1}^{N+H} \|q\|^2 \frac{1}{k} + \sum_{k>N+H} (\mathcal{E}_H(q))^2 \left(\frac{1}{k} - \frac{1}{k+N} \right) \\ &\leq \|q\|^2 \frac{H}{N} + (\mathcal{E}_H(q))^2 \sum_{k=N}^{\infty} \frac{N}{k(k+1)} = \|q\|^2 \frac{H}{N} + (\mathcal{E}_H(q))^2. \end{aligned}$$

If $q \in \ell^2(\Omega)$ with $\Omega(k) = (\log(e + |k|))^\beta$, then by (4.64) $\mathcal{E}_M(q) \leq \frac{\mathcal{E}_M^\Omega(q)}{(\log(e+M))^\beta}$. Therefore, (4.75) with $H = \sqrt{N}$ implies (4.76). \square

Now (4.67) and (4.69) follow immediately from (4.71) and, respectively, (4.75) and (4.76).

Lemma 22. *If $1 \leq a < 2$ and $q \in \ell^2(\Omega)$ with $\Omega(k) = 1 + |k|^\delta$, $\delta > 0$ then*

$$(4.77) \quad \sigma_1(a, N) + \sigma_2(a, N) \lesssim \|q\|^a \frac{H}{N^{a/2}} + (\mathcal{E}_H^\Omega(q))^a N^{1-\delta a - \frac{a}{2}}, \quad 0 < H < \frac{N}{2}.$$

Proof. In view of (4.64), $\mathcal{E}_M(q) \leq \mathcal{E}_M^\Omega(q)/\Omega(M) = \mathcal{E}_M^\Omega(q)/M^\delta$, so (4.73) implies

$$\sigma_1(a, N) \lesssim N^{-\frac{a}{2}} \sum_{k=0}^{N-H} \frac{(\mathcal{E}_{N+1-k}^\Omega(q))^a}{(N+1-k)^{a\delta}} + N^{-\frac{a}{2}} \sum_{N-H+1}^N \|q\|^2 + \frac{(\mathcal{E}_N^\Omega(q))^a}{N^{a\delta}} N^{1-\frac{a}{2}}.$$

Therefore, taking into account that $\mathcal{E}_{N+1-k}^\Omega(q) \leq \mathcal{E}_H^\Omega(q)$ for $0 \leq k \leq N-H$, we obtain

$$(4.78) \quad \sigma_1(a, N) \lesssim \|q\|^a \cdot \frac{H}{N^{a/2}} + (\mathcal{E}_H^\Omega(q))^a N^{1-a\delta-\frac{a}{2}}$$

because $\sum_{k=0}^{N-H} \frac{1}{(N+1-k)^{a\delta}} \lesssim \sum_{s=1}^N \frac{1}{s^{a\delta}} \lesssim N^{1-a\delta}$.

Next we estimate σ_2 in an analogous way. From (4.74) it follows that

$$\begin{aligned} \sigma_2(N) &= \sum_{N+1}^{N+H} \|q\|^2 \frac{1}{N^{a/2}} + \sum_{N+H+1}^{\infty} (\mathcal{E}_{k-N}(q))^a \left(\frac{1}{k} - \frac{1}{N+k} \right)^{\frac{a}{2}} + (\mathcal{E}_N(q))^a N^{1-a\delta-\frac{a}{2}} \\ &\leq \|q\|^a \cdot \frac{H}{N^{a/2}} + (\mathcal{E}_H^\Omega(q))^a \sum_{N+H+1}^{\infty} \frac{1}{(k-N)^{a\delta}} \left(\frac{1}{k} - \frac{1}{N+k} \right)^{\frac{a}{2}} + (\mathcal{E}_N^\Omega(q))^a N^{1-a\delta+\frac{a}{2}}. \end{aligned}$$

Since $\mathcal{E}_N^\Omega(q) \leq \mathcal{E}_H^\Omega(q)$ and

$$\begin{aligned} &\sum_{N+H+1}^{\infty} \frac{1}{(k-N)^{a\delta}} \left(\frac{1}{k} - \frac{1}{N+k} \right)^{\frac{a}{2}} = \sum_{N+1}^{2N} \frac{1}{(k-N)^{a\delta}} \frac{N^{\frac{a}{2}}}{k^{\frac{a}{2}}(k+N)^{\frac{a}{2}}} \\ &+ \sum_{2N+1}^{\infty} \frac{1}{(k-N)^{a\delta}} \frac{N^{\frac{a}{2}}}{k^{\frac{a}{2}}(k+N)^{\frac{a}{2}}} \lesssim N^{-\frac{a}{2}} \sum_{s=1}^N \frac{1}{s^{a\delta}} + N^{\frac{a}{2}} \sum_N^{\infty} \frac{1}{s^{a\delta+a}} \lesssim N^{1-a\delta-\frac{a}{2}}, \end{aligned}$$

we obtain

$$(4.79) \quad \sigma_2(a, N) \lesssim \|q\|^a \cdot \frac{H}{N^{a/2}} + (\mathcal{E}_H^\Omega(q))^a N^{1-a\delta-\frac{a}{2}}.$$

Now, (4.78) and (4.79) imply (4.77). \square

Finally, (4.71) and (4.77) imply (4.68), which completes the proof of Proposition 19. \square

5. THE CASE $v \in H_{per}^{-1}$, $S_N - S_N^0 : L^a \rightarrow L^b$, $b < \infty$.

1. Our main result in this section is the following.

Theorem 23. *Suppose S_N , S_N^0 are the spectral projections defined by (1.33) for the Hill operators $L_{bc}(v)$ and L_{bc}^0 subject to the boundary conditions $bc = Per^\pm$ or Dir . Let $v \in H_{per}^{-1}$, $v = Q'$, $Q \in L^2([0, \pi])$, and let $q = (q(k))_{k \in 2\mathbb{Z}}$ and $\tilde{q} = (\tilde{q}(m))_{m \in \mathbb{N}}$ be, respectively, the sequences of the Fourier coefficients of Q about the o.n.b. $\{e^{ikx}, k \in 2\mathbb{Z}\}$ and $\{\sqrt{2} \sin mx, m \in \mathbb{N}\}$. If*

$$(5.1) \quad 1 < a \leq b < \infty \quad \text{with} \quad \delta := 1/2 - (1/a - 1/b) > 0,$$

then

$$(5.2) \quad \|S_N - S_N^0 : L^a \rightarrow L^b\| \lesssim N^{-\tau} + \begin{cases} \mathcal{E}_N(q) & \text{if } bc = Per^\pm, \\ \mathcal{E}_N(\tilde{q}) & \text{if } bc = Dir, \end{cases}$$

where $\tau = \delta$ in the case $1 < a < 2 < b < \infty$, and otherwise one may take any τ such that

$$\tau < \begin{cases} 1 - 1/a & \text{if } 1 < a \leq b \leq 2, \\ 1/b & \text{if } 2 \leq a \leq b < \infty. \end{cases}$$

Remark. This theorem (quoted as Proposition 16 in [10]) is an important element in the proof of our Criterion for basisness in L^p , $p \neq 2$, of the system of root functions in the case of Hill operators with singular potentials.

Proof. By (1.34), $S_N - S_N^0 = T_N + B_N$, where T_N and B_N are the operators defined by (1.35) and (1.36). If $1 < a < 2 < b < \infty$, then Propositions 24 and 25 below imply (5.2) with $\tau = \delta$.

If (5.1) holds but $a < 2 < b$ fails, then

$$\text{either (i) } 1 < a \leq b \leq 2, \quad \text{or (ii) } 2 \leq a \leq b < \infty.$$

We set, respectively, $a_1 = a - \varepsilon$, $b_1 = 2 + \varepsilon$ in the case (i), and $a_1 = 2 - \varepsilon$, $b_1 = b + \varepsilon$ in the case (ii). Then, for small enough $\varepsilon > 0$, we have

$$1 < a_1 < 2 < b_1 < \infty, \quad \delta_1 = 1/2 - (1/a_1 - 1/b_1) > 0.$$

Since $\|S_N - S_N^0 : L^a \rightarrow L^b\| \leq \|S_N - S_N^0 : L^{a_1} \rightarrow L^{b_1}\|$, it follows that

$$\|S_N - S_N^0 : L^a \rightarrow L^b\| \lesssim N^{-\delta_1} + \begin{cases} \mathcal{E}_N(q) & \text{if } bc = Per^\pm, \\ \mathcal{E}_N(\tilde{q}) & \text{if } bc = Dir, \end{cases}$$

so (5.2) holds with $\tau = \delta_1$. But in both cases $\delta_1 = \delta_1(\varepsilon)$ is a monotone decreasing function of ε such that $\lim_{\varepsilon \rightarrow 0} \delta_1(\varepsilon) = \begin{cases} 1 - 1/a & \text{if } 1 < a \leq b \leq 2, \\ 1/b & \text{if } 2 \leq a \leq b < \infty. \end{cases}$

This completes the proof up to Propositions 24 and 25 below. \square

2. Next we estimate the norms $\|T_N : L^a \rightarrow L^b\|$.

Proposition 24. *If $v \in H_{per}^{-1}$, then for $1 < a < 2 < b < \infty$ with*

$$(5.3) \quad \delta = 1/2 - (1/a - 1/b) > 0$$

$$(5.4) \quad \|T_N : L^a \rightarrow L^b\| \lesssim N^{-\delta} + \begin{cases} \mathcal{E}_N(q) & \text{if } bc = Per^\pm, \\ \mathcal{E}_N(\tilde{q}) & \text{if } bc = Dir. \end{cases}$$

Proof. As in Section 3.3, we obtain the matrix representation of the operator T_N after integration over $\partial\Pi_N$. If T_{mk} is its matrix representation with respect to the basis $\{u_k, k \in \Gamma_{bc}\}$ of eigenfunctions of the free operator L_{bc}^0 , then $T_{mk} = 0$ for $(m, k) \notin X$, where $X = X(N)$ is defined in (3.17) or (3.24).

By the Hausdorff-Young theorems,

$$(5.5) \quad \|T_N : L^a \rightarrow L^b\| \leq 4\tilde{\tau}, \quad \tilde{\tau} = \|\tilde{T}_N : \ell^\alpha \rightarrow \ell^\beta\|$$

where

$$(5.6) \quad 1/a + 1/\alpha = 1, \quad 1/b + 1/\beta = 1$$

and the operator \tilde{T}_N is defined by its matrix, respectively given by (3.16) if $bc = Per^\pm$ and (3.23) if $bc = Dir$. Further we provide details only in the case $bc = Per^\pm$ because the proof for $bc = Dir$ is the same.

By duality

$$(5.7) \quad \tilde{\tau} \leq \sup \left\{ \sum_{(k,m) \in X_N} \frac{|V(m-k)|}{|m^2 - k^2|} |f(k)| |g(m)| : \|f|_{\ell^\alpha}\| = 1, \|g|_{\ell^b}\| = 1 \right\}.$$

Therefore, in view of (1.17) we need to evaluate

$$(5.8) \quad \tau(f, g) = \sum_{(k,m) \in X_N} \frac{|q(m-k)|}{|m+k|} |f(k)| |g(m)|.$$

We set

$$(5.9) \quad \Delta_N = \{(k, m) \in X_N : |m - k| \leq N\}, \quad \Delta_N^c = X_N \setminus \Delta_N,$$

and analyze the corresponding partial sums of $\tau(f, g)$.

Since $|m + k| \geq N$ on Δ_N , it follows that

$$(5.10) \quad \sum_{(k,m) \in \Delta_N} \leq \frac{4}{N} \sum_{j=1}^N |q(j)| \left(\sum_{\ell_j} |f(k)| |g(m)| \right),$$

where

$$(5.11) \quad \ell_j = \{(k, m) \in X_N : |m - k| = j\}.$$

If

$$(5.12) \quad \frac{1}{\alpha} + \frac{1}{b} + \frac{1}{\gamma} = 1$$

(so, by (5.3), $\frac{1}{\gamma} = \frac{1}{a} - \frac{1}{b} < \frac{1}{2}$), then by the triple Hölder inequality

$$(5.13) \quad \sum_{\ell_j} |f(k)| \cdot |g(m)| \cdot 1 \leq \|f|_{\ell^\alpha}\| \cdot \|g|_{\ell^b}\| \cdot (\text{card } \ell_j)^{1/\gamma} \leq (4j)^{1/\gamma},$$

and by the Cauchy inequality

$$(5.14) \quad \sum_1^N |q(j)| j^{1/\gamma} \leq \|q\| \left(\sum_1^N j^{2/\gamma} \right)^{1/2} \sim \|q\| \cdot N^{1/\gamma+1/2}.$$

With extra-factor $1/N$ in (5.10) these inequalities imply that

$$(5.15) \quad \sum_{\Delta_N} \leq C(\gamma) N^{-\delta}, \quad \delta = \frac{1}{2} - \frac{1}{\gamma}.$$

To estimate $\sum_{\Delta_N^c}$ we choose positive p, q, r with $p+q+r = 1$ in the following way:

$$(5.16) \quad p = t/2, \quad q = t(1/a - 1/2), \quad r = t(1/2 - 1/b)$$

with

$$(5.17) \quad 1/2 < 1/t = (1/a - 1/b) + 1/2 < 1.$$

Then

$$(5.18) \quad \sum_{\Delta_N^c} \frac{|q(m-k)|}{|m+k|^p} \cdot \frac{|f(k)|}{|m+k|^p} \cdot \frac{|g(m)|}{|m+k|^r} \\ \leq \left(\sum_{\Delta_N^c} \frac{|q(m-k)|^2}{|m+k|^{2p}} \right)^{1/2} \left(\sum_{\Delta_N^c} \frac{|f(k)|^2}{|m+k|^{2q}} \cdot \frac{|g(m)|^2}{|m+k|^{2r}} \right)^{1/2}.$$

With $|m-k| > N$ on Δ_N^c the first factor in the right-hand side of (5.18) does not exceed

$$(5.19) \quad 8\mathcal{E}_N(q) \cdot \left(\sum_1^\infty \frac{1}{j^{2p}} \right)^{1/2} = C(p)\mathcal{E}_N(q) < \infty.$$

In the second factor we want to make k and m independent; we can achieve this on four subsets of Δ_N^c separately, where

$$(5.20) \quad \Delta_N^c = F_1^+ \cup F_1^- \cup F_2^+ \cup F_2^-$$

with

$$(5.21) \quad F_j^\pm = \{(k_1, k_2) \in \Delta_N^c : |k_j| \leq N, \pm k_{j'} > 0\}.$$

For $(k, m) \in F_1^+$ we have $|k| \leq N$ and $m \geq N+1$; then either $|m+k| = m+k \geq m-N \geq 1$ or $m+k \geq N+1+k \geq N+1$. Therefore,

$$\sum_{F_1^+} \frac{|f(k)|^2}{|m+k|^{2q}} \cdot \frac{|g(m)|^2}{|m+k|^{2r}} \leq \sum_{F_1^+} \frac{|f(k)|^2}{|N+1+k|^{2q}} \cdot \frac{|g(m)|^2}{|m-N|^{2r}} \\ \leq \sum_{i=1}^\infty \frac{|f(-1-N+i)|^2}{i^{2q}} \cdot \sum_{j=1}^\infty \frac{|g(N+j)|^2}{j^{2r}} := \mathcal{F} \cdot \mathcal{G}.$$

Each of these two factors \mathcal{F}, \mathcal{G} is estimated by the Hölder inequality, respectively with parameters $\alpha/2, \tilde{\alpha}$ and $b/2, \tilde{b}$, i.e.,

$$\frac{2}{\alpha} + \frac{1}{\tilde{\alpha}} = 1, \quad \frac{2}{b} + \frac{1}{\tilde{b}} = 1.$$

This choice, together with (5.16) and (5.17) guarantees that

$$2q\tilde{\alpha} > 1, \quad 2r\tilde{b} > 1,$$

so the first factor does not exceed

$$\mathcal{F} \leq \left(\sum_j |f(-1-N+i)|^\alpha \right)^{2/\alpha} \cdot \left(\sum_i (1/i)^{2q\tilde{\alpha}} \right)^{1/\tilde{\alpha}} < \infty.$$

The same argument with $2r\tilde{b} > 1$ shows that

$$\mathcal{G} \leq C(g) \cdot \left(\sum_j (1/j)^{2r\tilde{b}} \right)^{1/\tilde{b}} < \infty.$$

The other sums over F_j^\pm could be estimated in an analogous way. This shows that the sum in (5.18) does not exceed $C(a, b) \cdot \mathcal{E}_N(q)$, so together with (5.15) we obtain for the form $\tau(f, g)$ that

$$\tau(f, g) \leq C(a, b) \left(N^{-\delta} + \mathcal{E}_N(q) \right).$$

This implies (5.4), which completes the proof. \square

3. Finally, we estimate the norms $\|B_N : L^a \rightarrow L^b\|$.

Proposition 25. *If*

$$(5.22) \quad 1 \leq a < 2 < b \leq \infty, \quad 1/a - 1/b < 1,$$

then

$$(5.23) \quad \|B_N : L^a \rightarrow L^b\| \lesssim \frac{\|q\|^2}{N} + \begin{cases} (\mathcal{E}_{\sqrt{N}}(q))^2 & \text{if } bc = \text{Per}^\pm, \\ (\mathcal{E}_{\sqrt{N}}(\tilde{q}))^2 & \text{if } bc = \text{Dir}. \end{cases}$$

Proof. We provide details only in the case $bc = \text{Per}^\pm$ because the proof for $bc = \text{Dir}$ is the same.

By (4.53), as in the proof of Proposition 17, it follows that

$$\|B_N : L^a \rightarrow L^b\| \leq \int_{\Lambda_N \cup \Lambda_N^-} \sum_{m=2}^{\infty} \|R_\lambda^0 (V R_\lambda^0)^m : L^a \rightarrow L^b\| dy.$$

By (4.23), $R_\lambda^0 (V R_\lambda^0)^m = K_\lambda (K_\lambda V K_\lambda)^m K_\lambda$, so we have

$$(5.24) \quad \|R_\lambda^0 (V R_\lambda^0)^m\|_{L^a \rightarrow L^b} \leq \|K_\lambda\|_{L^a \rightarrow L^2} \|K_\lambda V K_\lambda\|_{L^2 \rightarrow L^2}^m \|K_\lambda\|_{L^2 \rightarrow L^b}.$$

Recall that K_λ is defined by (4.24) as a multiplier operator in the sequence spaces of Fourier coefficients. If $f \in L^a$ and (f_k) is its sequence of Fourier coefficients about $\{u_k(x)\}$ – one of our canonical o.n.b. (1.11), (1.12) – then $K_\lambda f = \sum_k \frac{1}{(\lambda - k^2)^{1/2}} f_k u_k(x)$. By Hausdorff-Young Theorem $(f_k) \in \ell^\alpha$ with $\frac{1}{a} + \frac{1}{\alpha} = 1$, and $\|(f_k)\|_{\ell^\alpha} \leq 2\|f\|_{L^a}$. The Hölder inequality implies (compare with (2.11)) that

$$\|K_\lambda : L^a \rightarrow L^2\| \lesssim \left(\sum_k \frac{1}{|\lambda - k^2|^{\frac{a}{2-a}}} \right)^{\frac{2-a}{2a}}, \quad 1 \leq a < 2.$$

By duality argument, $\|K_\lambda : L^2 \rightarrow L^b\| = \|K_\lambda : L^\beta \rightarrow L^2\|$, where $\frac{1}{\beta} + \frac{1}{b} = 1$, so it follows that

$$\|K_\lambda : L^2 \rightarrow L^b\| \lesssim \left(\sum_k \frac{1}{|\lambda - k^2|^{\frac{\beta}{2-\beta}}} \right)^{\frac{2-\beta}{2\beta}}, \quad 2 < b \leq \infty, \quad \frac{1}{\beta} = 1 - \frac{1}{b}.$$

Therefore, in view of (6.2) we have

$$\|K_\lambda\|_{L^a \rightarrow L^2} \lesssim A^{1/2} \left(\lambda, \frac{a}{2-a} \right), \quad \|K_\lambda\|_{L^2 \rightarrow L^b} \lesssim A^{1/2} \left(\lambda, \frac{\beta}{2-\beta} \right).$$

Since the Hilbert-Schmidt norm dominates the L^2 -norm, (5.24) and the above formulas imply that

$$\sum_{m=2}^{\infty} \|R_\lambda^0 (V R_\lambda^0)^m\|_{L^a \rightarrow L^b} \leq S_1(\lambda),$$

where

$$S_1(\lambda) := A^{1/2} \left(\lambda, \frac{a}{2-a} \right) A^{1/2} \left(\lambda, \frac{\beta}{2-\beta} \right) \sum_{m=2}^{\infty} \|K_\lambda V K_\lambda\|_{HS}^m.$$

As in the proof of Proposition 17, by (4.27) one can easily see that $\int_{\Lambda_N^-} S_1(\lambda) dy \leq \int_{\Lambda_N} S_1(\lambda) dy$. Therefore,

$$\|B_N\|_{L^a \rightarrow L^b} \lesssim \int_{\Lambda_N} S_1(\lambda) dy.$$

In view of (4.55), for large enough N we have $\|K_\lambda V K_\lambda\|_{HS} < 1/2$ for $\lambda \in \Lambda_N$, so

$$\sum_{m=2}^{\infty} \|K_\lambda V K_\lambda\|_{HS}^m \leq 2 \|K_\lambda V K_\lambda\|_{HS}^2, \quad N \geq N_*.$$

Thus, by (4.30) and (4.32), we obtain $S_1(\lambda) \leq \Phi_N(y)$ with

$$(5.25) \quad \Phi_N(y) := A^{\frac{1}{2}} \left(\lambda, \frac{a}{2-a} \right) A^{\frac{1}{2}} \left(\lambda, \frac{\beta}{2-\beta} \right) \psi_N(y), \quad \lambda = N^2 + N + iy.$$

In view of the above formulas,

$$(5.26) \quad \|B_N\|_{L^a \rightarrow L^b} \lesssim \int_{\mathbb{R}} \Phi_N(y) dy \lesssim I_1 + I_2 + I_3, \quad N \geq N_*,$$

where

$$I_1 = \int_{|y| \leq N} \Phi_N(y) dy, \quad I_2 = \int_{N \leq |y| \leq N^2} \Phi_N(y) dy, \quad I_3 = \int_{|y| \geq N^2} \Phi_N(y) dy.$$

Next we estimate these integrals.

If $|y| \leq N$ then by (4.56), (4.57) and (6.26) we have

$$a_N(y) \lesssim \frac{\log N}{N}, \quad b_N(y) \lesssim \frac{1}{N^2}, \quad A(\lambda, r) \lesssim \frac{1}{N} \quad \forall r \geq 1.$$

Therefore, from (4.33), (4.34) and (5.25) it follows that

$$\Phi_N(y) \lesssim \frac{1}{N} \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right) + (\mathcal{E}_{4N}(q))^2 \frac{\log N}{N^2},$$

so we obtain

$$(5.27) \quad I_1 \lesssim \frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 + (\mathcal{E}_{4N}(q))^2 \frac{\log N}{N} \lesssim \frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2.$$

Next we estimate I_2 . If $\lambda = N^2 + N + iy$ with $N \leq |y| \leq N^2$, then (4.56), (4.57) and (6.26) imply that

$$a_N(y) \lesssim \frac{1}{N} \log \left(1 + \frac{N^2}{|y|} \right), \quad b_N(y) \lesssim \frac{1}{N|y|}, \quad A(\lambda, r) \lesssim N^{-\frac{1}{r}} |y|^{-1+\frac{1}{r}}.$$

Therefore,

$$A^{1/2} \left(\lambda, \frac{a}{2-a} \right) A^{1/2} \left(\lambda, \frac{\beta}{2-\beta} \right) \lesssim N^{1-\frac{1}{a}-\frac{1}{\beta}} |y|^{-2+\frac{1}{a}+\frac{1}{\beta}} = \frac{1}{N} N^\gamma |y|^{-\gamma},$$

where

$$0 < \gamma := 2 - 1/a - 1/\beta = 1 - 1/a + 1/b < 1$$

due to (5.22).

Now from (4.33), (4.34) and (5.25) we obtain

$$\Phi_N(y) \lesssim N^\gamma |y|^{-1-\gamma} \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right) + (\mathcal{E}_{4N}(q))^2 \frac{|y|^{-\gamma}}{N^{2-\gamma}} \log \left(1 + \frac{N^2}{|y|} \right),$$

Since $\int_N^{N^2} y^{-1-\gamma} dy \lesssim N^{-\gamma}$ and (with the change of variable $t = N^2/y$)

$$\int_N^{N^2} y^{-\gamma} \log(1 + N^2/y) dy = N^{2-2\gamma} \int_0^N \frac{1}{t^{2-\gamma}} \log(1+t) dt \lesssim N^{2-2\gamma},$$

it follows that

$$(5.28) \quad I_2 \lesssim \|q\|^2/N + (\mathcal{E}_{\sqrt{N}}(q))^2 + N^{-\gamma} (\mathcal{E}_{4N}(q))^2.$$

Finally, we estimate I_3 . For $\lambda = N^2 + N + iy$ with $|y| \geq N^2$ we have by (4.56), (4.57) and (6.26) that

$$a_N(y) \lesssim \frac{1}{|y|^{1/2}}, \quad b_N(y) \lesssim \frac{1}{|y|^{3/2}}, \quad A(\lambda, r) \lesssim |y|^{-1+\frac{1}{2r}}.$$

Therefore,

$$A^{1/2} \left(\lambda, \frac{a}{2-a} \right) A^{1/2} \left(\lambda, \frac{\beta}{2-\beta} \right) \lesssim |y|^{-\frac{3}{2}+\frac{1}{2a}+\frac{1}{2\beta}} = |y|^{-\frac{1}{2}-\frac{\gamma}{2}},$$

so (4.33), (4.34) and (5.25) imply that

$$\Phi_N(y) \lesssim N^2 |y|^{-2-\gamma/2} \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right) + |y|^{-1-\gamma/2} (\mathcal{E}_{4N}(q))^2.$$

Now, integrating over $|y| \geq N^2$, we obtain

$$(5.29) \quad I_3 \lesssim \left(\frac{\|q\|^2}{N} + (\mathcal{E}_{\sqrt{N}}(q))^2 \right) N^{-\gamma} + (\mathcal{E}_{4N}(q))^2 N^{-\gamma}.$$

The estimates (5.27), (5.28) and (5.29) yield (5.23), which completes the proof. \square

6. APPENDIX: AUXILIARY INEQUALITIES

In this section we collect a few inequalities that justify crucial steps in the proof of our main results and could be useful elsewhere.

The elementary inequality

$$(6.1) \quad (a+b)^\tau \leq 2^{\tau-1}(a^\tau + b^\tau), \quad a, b > 0, \quad \tau \geq 1,$$

will be used throughout the text often without any specification. Of course, (6.1) explains that for every fixed $\tau > 0$

$$(a+b)^\tau \sim a^\tau + b^\tau, \quad a, b > 0.$$

Next, for fixed $r \geq 1$, we analyze the behavior of the function

$$(6.2) \quad A(z, r) = \left(\sum_{k=0}^{\infty} \frac{1}{|z - k^2|^r} \right)^{1/r}, \quad r \geq 1, \quad z \in \mathbb{C}.$$

We need estimates of this function and its integrals on properly chosen contours in \mathbb{C} .

1. Horizontal lines $z = x + ih$, $x \in \mathbb{R}$.

Since $A(x + ih, r) = A(x + i|h|, r)$, we assume for simplicity of the writing that $h > 0$. If $z = x + ih$ then

$$(6.3) \quad \frac{1}{2}(|x - k^2| + h) \leq |z - k^2| \leq |x - k^2| + h$$

and

$$(6.4) \quad |z - k^2|^r \geq 2^{-r}(|x - k^2| + h)^r.$$

Therefore,

$$(6.5) \quad [A(x + ih, r)]^r \leq 2^r \sum_0^{\infty} \frac{1}{(|x - k^2| + h)^r} = 2^r(\sigma_0^r + \sigma_1^r),$$

where

$$(6.6) \quad \sigma_0^r = \sum_0^{2N} \frac{1}{(|x - k^2| + h)^r} \leq (2N + 1) \cdot h^{-r}$$

and

$$(6.7) \quad \sigma_1^r = \sum_{2N+1}^{\infty} \frac{1}{(|x - k^2| + h)^r}.$$

If

$$(6.8) \quad |x| \leq N^2 + N,$$

and $k \geq 2N + 1$, then

$$(6.9) \quad |x - k^2| + h \geq \frac{1}{2}(k^2 + h) \geq \frac{1}{4}(k + \sqrt{h})^2.$$

Indeed, if $x \leq 0$, then (6.9) is obvious. If $0 < x \leq N^2 + N$, then $|x - k^2| \geq k^2 - (N^2 + N) \geq \frac{1}{2}k^2$ because $k \geq 2N + 1$.

Therefore, if (6.8) holds, then

$$(6.10) \quad \sigma_1^r \leq \sum_{2N+1}^{\infty} \frac{4^r}{(k + \sqrt{h})^{2r}} \leq \int_{2N}^{\infty} \frac{4^r}{(\xi + \sqrt{h})^{2r}} d\xi = \frac{4^r}{2r-1} (2N + \sqrt{h})^{-2r+1}.$$

Now, in view of (6.5), (6.6) and (6.10), it follows that

$$A(x + ih, r) \leq 2(\sigma_0^r + \sigma_1^r)^{1/r} \lesssim \frac{N^{1/r}}{h} + \frac{1}{(N^2 + h)^{1-1/(2r)}} \lesssim \frac{1}{h} (N^{1/r} + h^{\frac{1}{2r}}).$$

The above inequalities imply the following.

Lemma 26. *If $|x| \leq N^2 + N$ and $h \geq N^2$, then*

$$(6.11) \quad A(x \pm ih, r) \lesssim h^{\frac{1}{2r}-1}$$

and

$$(6.12) \quad \int_{-\omega}^{N^2+N} (A(x \pm ih, r))^m dx \lesssim (N^2 + \omega) \cdot h^{-m(1-\frac{1}{2r})}, \quad m > 0.$$

If $m \geq 1$, then

$$(6.13) \quad \lim_{h \rightarrow \infty} \int_{-\omega}^{N^2+N} (A(x \pm ih, r))^m dx = 0.$$

2. Vertical lines $z = -\omega + iy$, $\omega \gg 1$.

In this case $|z - k^2| \sim k^2 + \omega + |y| \sim (k + \sqrt{\omega + |y|})^2$. Hence, for every fixed $r \geq 1$ we have

$$(6.14) \quad A^r \sim \sum_{k=0}^{\infty} \frac{1}{(k + \sqrt{\omega + |y|})^{2r}} \sim \int_0^{\infty} \frac{1}{(x + \sqrt{\omega + |y|})^{2r}} dx \lesssim \frac{1}{(\omega + |y|)^{r-\frac{1}{2}}}.$$

Therefore, the following holds.

Lemma 27. *For fixed $r \geq 1$, $\omega > 0$*

$$(6.15) \quad A(-\omega + iy, r) \lesssim \left(\frac{1}{\omega + |y|} \right)^{1-\frac{1}{2r}}.$$

Moreover, if $r > 1$ then

$$(6.16) \quad \int_{-\infty}^{\infty} A^2(-\omega + iy, r) dy \rightarrow 0 \quad \text{as } \omega \rightarrow \infty,$$

and if $r = 1$

$$(6.17) \quad \int_{-\infty}^{\infty} A^3(-\omega + iy, 1) dy \rightarrow 0 \quad \text{as } \omega \rightarrow \infty.$$

3. Now we analyze $A(z, r)$ on the vertical line

$$(6.18) \quad \Lambda_N = \{\lambda = N^2 + N + iy, y \in \mathbb{R}\}.$$

Since $A(N^2 + N + iy, r) = A(N^2 + N + i|y|, r)$, we assume for simplicity that $y \geq 0$.

One can easily see for $\lambda = N^2 + N + iy$ that

$$(6.19) \quad |\lambda - k^2| \sim |N^2 - k^2| + N + y.$$

Therefore, in view of (6.2)

$$(6.20) \quad A(\lambda, r) \sim \sigma_0 + \sigma_1 \quad \text{for } \lambda \in \Lambda_N,$$

where

$$(6.21) \quad (\sigma_0)^r = \sum_{k=0}^{2N} \frac{1}{(|N^2 - k^2| + N + y)^r}, \quad (\sigma_1)^r = \sum_{k=2N+1}^{\infty} \frac{1}{(|N^2 - k^2| + N + y)^r}.$$

Next we estimate $(\sigma_0)^r$. Since $|N^2 - k^2| = |N - k|(N + k) \sim |N - k|N$ if $0 \leq k \leq 2N$, it follows that

$$\begin{aligned} (\sigma_0)^r &\sim \sum_{j=0}^N \frac{1}{(Nj + N + y)^r} \sim \int_0^N \frac{1}{(N\xi + N + y)^r} d\xi. \\ &\sim \begin{cases} \frac{1}{N} \log \left(1 + \frac{N^2}{N+y} \right) & \text{if } r = 1, \\ \frac{1}{N} \left(\frac{1}{(N+y)^{r-1}} - \frac{1}{(N^2+N+y)^{r-1}} \right) & \text{if } r > 1. \end{cases} \end{aligned}$$

Therefore, by using Mean Value Theorem in the case $r > 1$, $y > N^2$, we obtain the following.

Lemma 28. *In the above notations,*

$$(6.22) \quad \sigma_0 \sim \frac{1}{N} \log \left(1 + \frac{N^2}{N+y} \right) \quad \text{if } r = 1,$$

and for $r > 1$

$$(6.23) \quad \sigma_0 \sim \begin{cases} \frac{1}{N^{1/r}} \cdot \frac{1}{(N+y)^{1-1/r}} & \text{if } 0 \leq y \leq N^2, \\ \frac{N^{1/r}}{y} & \text{if } y > N^2. \end{cases}$$

4. Now we estimate the sum $(\sigma_1)^r$ defined in (6.21). If $k \geq 2N + 1$ then

$$k^2 - N^2 + N + y \sim k^2 + y \sim (k + \sqrt{y})^2,$$

so

$$(\sigma_1)^r \sim \sum_{k=2N+1}^{\infty} \frac{1}{(k + \sqrt{y})^{2r}} \sim \int_{2N}^{\infty} \frac{1}{(\xi + \sqrt{y})^{2r}} d\xi \sim (2N + \sqrt{y})^{-2r+1}.$$

Therefore, we obtain

$$(6.24) \quad \sigma_1 \sim (N^2 + y)^{-1+\frac{1}{2r}}.$$

This, together with Lemma 28, (6.22) and (6.23), leads us to the following.

Lemma 29. *With notations (6.2), we have*

$$(6.25) \quad A(N^2 + N + iy, 1) \sim \begin{cases} \frac{\log N}{N} & \text{if } |y| \leq N, \\ \frac{1}{N} \log(1 + \frac{N^2}{|y|}) & \text{if } N \leq |y| \leq N^2, \\ \frac{1}{\sqrt{|y|}} & \text{if } |y| \geq N^2, \end{cases}$$

and, for $r > 1$,

$$(6.26) \quad A(N^2 + N + iy, r) \sim \begin{cases} \frac{1}{N} & \text{if } |y| \leq N, \\ N^{-\frac{1}{r}} |y|^{-1+\frac{1}{r}} & \text{if } N \leq |y| \leq N^2, \\ |y|^{-1+\frac{1}{2r}} & \text{if } |y| \geq N^2. \end{cases}$$

Proof. Indeed, (6.22) and (6.24) lead to (6.25), and (6.23) together with (6.24) imply (6.26). \square

The inequality (6.26) helps us to give estimates of $\int_{\Lambda_N} A^2(\lambda; r) dy$ from above. We have the following three cases:

- (i) $r > 2$;
- (i) $r = 2$;
- (iii) $1 < r < 2$.

In either case ($r > 1$),

$$(6.27) \quad \int_{|y| \leq N} A^2 dy \lesssim (1/N)^2 \cdot 2N \lesssim \frac{1}{N}$$

and

$$(6.28) \quad \int_{|y| \geq N^2} A^2(\lambda, r) dy \lesssim \int_{N^2}^{\infty} y^{-2+1/r} dy \lesssim N^{-2(1-1/r)}.$$

The integration over the interval $[N, N^2]$ is also easy but the result depends on r in an essential way. By (6.26), the middle line,

$$(6.29) \quad \int_{N \leq |y| \leq N^2} A^2(\lambda, r) dy \lesssim Y(N),$$

where

$$Y(N) := \int_N^{N^2} N^{-2/r} y^{-2+2/r} dy = \begin{cases} \frac{\log N}{N} & \text{if } r = 2; \\ \frac{r}{2-r} \left(N^{-2(1-\frac{1}{r})} - N^{-1} \right) & \text{if } r \neq 2. \end{cases}$$

Therefore

$$(6.30) \quad Y(N) \lesssim \begin{cases} \frac{1}{N} & \text{if } r > 2, \\ \frac{\log N}{N} & \text{if } r = 2, \\ N^{-2(1-\frac{1}{r})} & \text{if } r < 2. \end{cases}$$

Now the inequalities (6.27)–(6.30) imply the following.

Corollary 30. *If $1 < r < \infty$, then*

$$(6.31) \quad \int_{\Lambda_N} A^2(\lambda; r) dy \leq \begin{cases} \frac{1}{N} & \text{if } r > 2, \\ \frac{\log N}{N} & \text{if } r = 2, \\ N^{-2(1-\frac{1}{r})} & \text{if } r < 2. \end{cases}$$

Next we consider the case $r = 1$.

Corollary 31. *From (6.25) it follows that*

$$(6.32) \quad \int_{\Lambda_N} A^3(\lambda; 1) dy \lesssim \frac{1}{N}.$$

Proof. The integral over $[0, N]$ does not exceed $C \frac{(\log N)^3}{N^2}$, and the integral over $[N^2, \infty)$ is less than C/N . Next, with $w = 1 + N^2/y$, $dy = -\frac{N^2}{(w-1)^2} dw$ we estimate $\int_N^{N^2} A^3(\lambda; 1) dy$ by

$$(6.33) \quad \int_N^{N^2} \left(\frac{1}{N} \log(1 + N^2/y) \right)^3 dy = \frac{1}{N} \int_2^{1+N} \frac{(\log w)^3}{(w-1)^2} dw \lesssim \frac{1}{N}.$$

These estimates lead immediately to (6.32). \square

Notice however that

$$(6.34) \quad \int_{\Lambda_N} A^2(\lambda, 1) dy = \infty$$

because by the third line of (6.25) we have $A(\lambda, 1) \sim 1/y^{1/2}$ for $y > N^2$.

5. We need a few estimates of double sums as well.

Lemma 32. *The sequence*

$$(6.35) \quad A_N = \sum_{k=0}^N \sum_{m=N+1}^{\infty} \frac{1}{m^2 - k^2}, \quad N = 0, 1, 2, \dots$$

is bounded by $A_0 = \pi^2/6$. Moreover, A_N is monotone decreasing, and $\lim A_N = \pi^2/8$.

Proof. We have $A_0 = \sum_{m=1}^{\infty} \frac{1}{m^2} = \pi^2/6$ and

$$\begin{aligned} A_{N+1} - A_N &= \sum_{m=N+2}^{\infty} \frac{1}{m^2 - (N+1)^2} - \sum_{k=0}^N \frac{1}{(N+1)^2 - k^2} \\ &= \frac{1}{2(N+1)} \sum_{m=N+2}^{\infty} \left(\frac{1}{m - (N+1)} - \frac{1}{m + (N+1)} \right) \\ &\quad - \frac{1}{2(N+1)} \sum_{k=0}^N \left(\frac{1}{N+1-k} + \frac{1}{N+1+k} \right) \end{aligned}$$

$$= \frac{1}{2(N+1)} \left(\sum_{\nu=1}^{2(N+1)} \frac{1}{\nu} - \sum_{\nu=1}^{N+1} \frac{1}{\nu} - \sum_{\nu=N+1}^{2N+1} \frac{1}{\nu} \right) = -\frac{1}{4(N+1)^2}.$$

Therefore,

$$A_N = A_0 + \sum_{n=1}^N (A_n - A_{n-1}) = \frac{\pi^2}{6} - \frac{1}{4} \sum_{n=1}^N \frac{1}{n^2} \rightarrow \frac{\pi^2}{6} - \frac{1}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8}.$$

□

Lemma 33. *Let $H \in \mathbb{N}$, $0 < H < N$, and let*

$$\Delta_H = \{(k, m) : 0 \leq k \leq N, m \geq N+1, m-k \leq H\}.$$

Then

$$(6.36) \quad \sigma(N, H) = \sum_{\Delta_H} \frac{1}{m^2 - k^2} \leq \frac{H}{N}.$$

Proof. Observe that Δ_H consists of points (k, m) with integer coordinates lying inside the triangle bounded by the lines $k = N$, $m = N+1$, $m-k = H$. Moreover,

$$(6.37) \quad \Delta_H = \bigcup_{\nu=1}^H \ell_\nu, \quad \text{with} \quad \ell_\nu = \{(k, m) \in \Delta_H : m-k = \nu\}, \quad \#\ell_\nu = \nu,$$

so $\#\Delta_H = H(H+1)/2$. Since $m \geq N+1$, we have

$$\begin{aligned} \sigma &= \sum_{\Delta_H} \frac{1}{2m} \left(\frac{1}{m-k} + \frac{1}{m+k} \right) \leq \frac{1}{2(N+1)} \sum_{\Delta_H} \left(\frac{1}{m-k} + \frac{1}{m+k} \right) \\ &\leq \frac{1}{2(N+1)} \left(\sum_{\nu=1}^H \frac{1}{\nu} \#\ell_\nu + \frac{1}{N+1} \#\Delta_H \right) = \frac{H}{2(N+1)} + \frac{(H+1)H}{4(N+1)^2} \leq \frac{H}{N}, \end{aligned}$$

which completes the proof. □

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